# Supplier Mix Selection Under Quantity Discounts

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## 1 Introduction

The problem considered has the following features: a manufacturing company wants to buy some material that it needs to fulfill its demand over a planning horizon of $n$ periods, at minimum cost. Here each period may be a 4-hour production shift, or some other time interval. The deterministic demand for this material in each period is assumed to be known and given.

**Demand data:** $D_1, \dots, D_n$ in units, in periods $1, \dots, n$ (1)

Under the JIT (Just In Time) policy, units of material needed in each period are procured from the suppliers in that period itself to keep inventories low. We assume that this policy is being followed.

There are $m$ suppliers who can supply this material, however each supplier has a limited capacity, and can only supply upto the following quantities in each period.

**Supplier Capacity Data** $k_{ij}$ units Maximum $i$th supplier can supply in $j$th period, $i = 1$ to $m$, $j = 1$ to $n$. (2)

If the total amount of material procured in any period is less than the demand in that period, the difference between them is the shortage in that period. Shortages can occur, when they do, production is disrupted, the cost of which is considered high.

**Shortage Cost/Unit** $\alpha$\$ is cost/unit short in any period (a high positive number). (3)

Now we define the decision variables whose values are to be determined optimally. These are:

**Amounts Ordered** $x_{ij}$ units amount ordered from the $i$th supplier in $j$th period, $i = 1$ to $m$, $j = 1$ to $n$. (4)

**Shortages** $s_j$ units amount of shortage (unfulfilled demand) in $j$th period, $j = 1$ to $n$. (5)

| | | |
|---|---|---|
| **Procurement from Suppliers** | $y_i = x_{i1} + \ldots + x_{in}$ | total ordered from $i$th supplier, $i = 1$ to $m$. |

(6)

Now we will describe the supply cost data. The $i$th supplier's charges depend only on $y_i$, the total quantity delivered by him/her over the entire planning horizon, and are based on the following broad principle of cost discounting honored universally in the business world: "the higher the quantity purchased, the lower the per unit cost". Let

$$f_i(y_i) = \text{cost of } y_i \text{ units supplied by } i\text{th supplier, } i = 1 \text{ to } m.$$

Some ways of implementing the above cost discounting principle, discussed in the literature, lead to discontinuous cost functions. We will however consider two different realizations of this principle that have found good acceptance from practitioners and also lend themselves nicely to mutual negotiations between buyers and sellers, which are:

**Incremental Discounting Cost Principle**

Here the practical range of values of $y_i$, 0 to some upper limit, is broken up into disjoint intervals inside each of which the cost per each additional unit

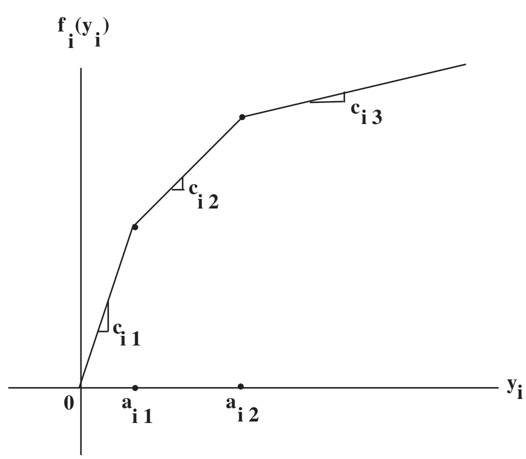


Figure 1: Piecewise linear concave cost function under incremental discounting.

remains constant, this constant strictly decreases as you move from one interval to the next one right of it. This data is displayed in Table 1 ($r$ is the total number of intervals into which the range of values of $y_i$ is broken up). Under this principle, $f_i(y_i)$ is a piecewise linear concave function (see Figure 1).

Table 1

| Interval of $y_i$ | Slope (cost/additional unit when $y_i$ in interval) |
|---|---|
| 0 to $a_{i1}$ | $c_{i1}$ |
| $a_{i1}$ to $a_{i2}$ | $c_{i2}$ |
| $\vdots$ | $\vdots$ |
| $a_{ir-1}$ to $a_{ir}$ | $c_{ir}$ |

$a_{ir}$ is upper limit for $y_i$, & $c_{i1} > c_{i2} > \ldots > c_{ir}$.

**Unit Discounting Cost Principle**

Here we assume that the cost per unit material ordered decreases linearly as $y_i$ increases, i.e.,

**Cost Data** $\delta_i - \gamma y_i$ = cost in \$/unit from $i$th supplier when quantity ordered is $y_i$, $i = 1$ to $m$. (7)

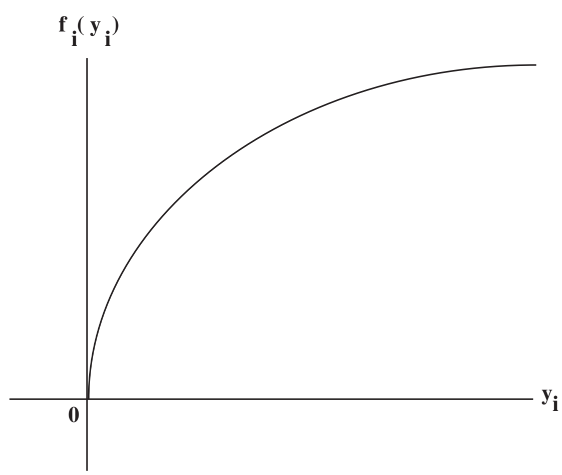


Figure 2: Concave quadratic cost function under unit discounting.

In this case we have $f_i(y_i) = y_i(\delta_i - \gamma_i y_i)$ where $\gamma_i > 0$, and $\delta_i > 2\gamma_i(k_{i1} + \ldots + k_{in}\})$ (this latter condition guarantees that the cost per unit ordered never becomes negative). In this case the cost function $f_i(y_i)$ is a concave quadratic function as shown in Figure 2.

Thus the input data for the problem consists of: demand data in eq. (1), supplier capacity data in eq. (2), shortage cost data in eq. (3), and the supplier cost data in Table 1 or eq. (7) depending on which cost function is preferred. The decision variables in the problem are the $x_{ij}, s_j, y_i$ defined in eqs. (4) (5), (6).

## 2 The Mathematical Model

The mathematical model for the problem is

$$\begin{array}{rrcll}
\text{Minimize} \quad g(x,s,y) = & \displaystyle\sum_{i=1}^{m} f_i(y_i) & + & \alpha(s_1 + \ldots + s_n) & \\
\text{subject to} & \displaystyle\sum_{i=1}^{m} x_{ij} + s_j & = & D_j \quad j = 1, \ldots, n & \\
& y_i - \displaystyle\sum_{j=1}^{n} x_{ij} & = & 0 \quad i = 1, \ldots, m & \qquad (8) \\
0 \leq x_{ij} \leq k_{ij} & i = 1, \ldots, m, & & j = 1, \ldots, n & \\
& s_j & \geq & 0 \quad j = 1, \ldots, n &
\end{array}$$

The objective function $g(x, s, y)$ is a concave function. Finding a global minimum for this problem is a hard problem because it may have local minima that are not global minima.

Next we will describe a heuristic approach for obtaining a good solution to this model with an estimate on the likely error.

## 3 A Heuristic Approach

We will first obtain a good initial feasible solution for the model using a lower bounding linear approximation for the objective function. This also yields lower and upper bounds for the global optimum objective value. If the difference between the bounds is small, this initial feasible solution can be accepted as an approximation to the optimum and the method terminated. Otherwise, a local search scheme is carried out to improve on the initial feasible solution.

**Step 1: Finding a good initial feasible solution**

1. For each $i = 1$ to $m$, select a practical upper bound $Y_i$ for the variable $y_i$ based on cost data, supplier's capacities, past experience, and any other practical information on the likely optimum value of $y_i$. Then in the interval of interest $0 \le y_i \le Y_i$, the linear function $c_i y_i$ where

$$c_i = f_i(Y_i)/Y_i$$

is a lower bound for $f_i(y_i)$ (see Figure 3) for each $i = 1$ to $m$.

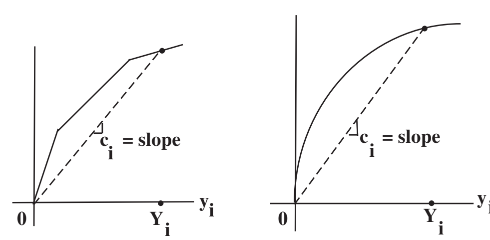


Figure 3: Lower bounding linear function for $f_i(y_i)$ for values of $y_i$ in the interval of interest $[0, Y_i]$.

2. Rearrange the suppliers in increasing order of $c_i$. After this permutation, we will have $c_1 \le c_2 \le \ldots \le c_m$.

   We will now minimize the linear objective function $\sum_{i=1}^{m} c_i y_i + \alpha \sum_{j=1}^{n} s_j$ subject to the constraints in (8). Since $c_i y_i = c_i \sum_{j=1}^{n} x_{ij}$, $c_i$ is the cost coefficient associated with $x_{ij}$ in this problem. This is a special problem for which an optimum solution can be found easily by this simple procedure: in the following 2-dimensional array, in each column $j$ representing period $j$ allocate the largest possible nonnegative value to the variables $x_{ij}$ i the order $i = 1$ to $m$ (top to bottom) taking the upper bound on it in (8) and the remaining demand in this column into account.

**Table 2:** Array for finding an initial feasible solution. Rows rearranged so that $c_1 \leq \ldots \leq c_m$. Fill demand to the maximum extent possible in each column separately in the order from top to bottom, noting $x_{ij} \leq k_{ij}$. At the end make the $s_j$ in that column equal to any left over demand.

| Period $\rightarrow$ | $j = 1$ | 2 | $\ldots$ | n |
|---|---|---|---|---|
| Supplier $i = 1$ | $x_{11}$ $c_1$ | $x_{12}$ $c_1$ | $\ldots$ $\ldots$ | $x_{1n}$ $c_1$ |
| 2 | $x_{21}$ $c_2$ | $x_{22}$ $c_2$ | $\ldots$ $\ldots$ | $x_{2n}$ $c_2$ |
| $\vdots$ | | | | |
| $m$ | $x_{m1}$ $c_m$ | $x_{m2}$ $c_m$ | $\ldots$ $\ldots$ | $x_{mn}$ $c_m$ |
| Shortage | $s_1$ $\alpha$ | $s_2$ $\alpha$ | $\ldots$ $\ldots$ | $s_n$ $\alpha$ |
| Demand | $D_1$ | $D_2$ | $\ldots$ | $D_n$ |

Let $(\bar{x} = (\bar{x}_{ij}), \bar{s} = (\bar{s}_{ij}))$ denote the solution obtained by this procedure. This is the initial feasible solution for (8). For $i = 1$ to $m$, define $\bar{y} = (\bar{y}_i)$, where $\bar{y}_i = \sum_{j=1}^{n} \bar{x}_{ij}$. Then

| | |
|---|---|
| $\sum_{i=1}^{m} c_i \bar{y}_i + \alpha \sum_{j=1}^{n} \bar{s}_j$ | is a lower bound for the optimum objective value in (8). |
| $g(\bar{x}, \bar{s}, \bar{y})$ | is an upper bound for the optimum objective value in (8). |

If the difference between the two bounds is small, accept $(\bar{x}, \bar{s}, \bar{y})$ as the approximate optimum solution and terminate. Otherwise go to Step 2.

**Step 2: Local search to move to a better solution**

Let $(\hat{x} = (\hat{x}_{ij}), \hat{s} = (\hat{s}_j), \hat{y} = (\hat{y}_i))$ be the present feasible solution. Look for a pair of suppliers $p$ and $q$ satisfying the following two conditions.

**(i)** $f'_p(\hat{y}_p) - f'_q(\hat{y}_q) < 0$. In the incremental discounting case $f'_p(\hat{y}_p)$ is the right derivative, and $f'_q(\hat{y}_q)$ is the left derivative here. Also, this difference can be approximated by $f_p(\hat{y}_p + 1) - f_p(\hat{y}_p) - (f_q(\hat{y}_q) - f_q(\hat{y}_q - 1))$.

**(ii)** It is possible to simultaneously increase $y_p$ above $\hat{y}_p$, and decrease $y_q$ below $\hat{y}_q$ (this means that there is at least one $j$ such that $\hat{x}_{pj} < k_{pj}$ and $\hat{x}_{qj} > 0$).

If a pair of suppliers $p$ and $q$ satisfying the above conditions are found, it is possible to get a better solution than the present $(\hat{x}, \hat{s}, \hat{y})$ by doing the following for each $j = 1$ to $n$:

replace $\hat{x}_{pj}$ by $\hat{x}_{pj} + \Delta$ and $\hat{x}_{qj}$ by $\hat{x}_{qj} - \Delta$ where $\Delta = \min\{k_{pj} - \hat{x}_{pj}, \hat{x}_{qj}\}$.

Repeat this step with the new solution obtained and continue as often as possible. Each of these steps decreases the objective value strictly.

When we reach a feasible solution where thre is no possibility of carrying out this local search step, that feasible solution is a local minimum for (8), and the heuristic algorithm terminates with the final feasible solution as an approximate optimum for (8).

## 4 How to Use a Storage Buffer

The model in (8) is based on the assumption that there is to be no storage of material from one period to the next.

Now, suppose there is a storage buffer available with the following capacity.

| | | | | |
|---|---|---|---|---|
| **Buffer Capacity** | $B$ | units = | maximum number of units that can be stored, each for any number of periods. | (9) |

When such a buffer is available, shortage in any period could be reduced by building up some stock in the buffer during earlier periods of excess supply. We now discuss an approach for utilizing this buffer to reduce the total cost.

In this case, in each period, we may buy material not only to meet demand in that period, but also to build up stock in the buffer for use in later periods when the supply may be short. The algorithm for this case will consist of three steps. In Step 0, we determine how much to buy in each period to make the shortage cost component of the total cost as small as possible. Then in Steps 1, 2 (which are similar to Steps 1, 2 of Section 3) we minimize the purchase cost component of the total cost.

**Step 0: Finding how much to buy in each period**

For $j = 1$ to $n$, let

$k_j = \sum_{I=1}^{m} k_{ij} =$ total amount of material available from all suppliers in period $j$.

The decision variables in the model for this step are

$\xi_j =$ amount of material purchased in period $j$ from all suppliers to meet demand in period $j$

$\eta_j =$ amount of material withdrawn from buffer in period $j$ to meet the demand in period $j$

$s_j =$ amount of shortage (unfulfilled demand) in period $j$ (same variable as in eq. (5))

$u_j =$ amount of material purchased in period $j$ from all suppliers for storing in buffer

$v_j =$ amount of material in storage in the buffer at the end of period $j$.

Then the model for minimizing the shortage cost component of the total cost is

$$
\begin{array}{rcll}
\text{Minimize} & & \displaystyle\sum_{j=1}^{n} s_j & \\
\text{subject to} \quad \eta_1 & = & 0 & \\
\xi_j + \eta_j + s_j & = & D_j & j = 1, \ldots, n \\
\xi_j + u_j & \leq & k_j & j = 1, \ldots, n \\
v_{j-1} - \eta_j & \geq & 0 & j = 2, \ldots, n \\
v_1 - u_1 & = & 0 & \\
v_j - (v_{j-1} + u_j - \eta_j) & = & 0 & j = 2, \ldots, n \\
v_j & \leq & B & j = 1, \ldots, n-1 \\
v_n & = & 0 & \\
\xi_j, \eta_j, s_j, u_j, v_j & \geq & 0 & \text{for all } j.
\end{array}
\tag{10}
$$

The reason for each of the constraints in this model can easily be understood.

This is a linear program (LP) with an optimum solution which can be found by solving it with any LP software. Let

$$(\tilde{\xi} = (\tilde{\xi}_j), \tilde{\eta} = (\tilde{\eta}_j), \tilde{s} = (\tilde{s}_j), \tilde{u} = (\tilde{u}_j), \tilde{v} = (\tilde{v}_j))$$

be an optimum solution of this LP. For $j = 1$ to $n$, let

$$\tilde{D} = \tilde{\xi}_j + \tilde{u}_j.$$

Then $\tilde{D}_j$ is the amount to be purchased from all suppliers put together in period $j$, $j = 1$ to $n$; and we know that it is feasible to purchase these quantities without encountering any shortages.

**Step 1: Finding a good initial feasible solution**

In this step, which is similar to Step 1 in the approach discussed in Section 3, we find an initial feasible solution for determining how to allocate the amount $\tilde{D}_j$ to be purchased in period $j$, $j = 1$ to $n$, among the various suppliers.

Determine the constants $c_i$ and rearrange them so that $c_1 \leq c_2 \leq \ldots \leq c_m$ as in Step 1 of Section 3.

**Table 3:** Array for finding an initial feasible solution. Rows rearranged so that $c_1 \leq \ldots \leq c_m$. For $j = 1$ to $n$, in column $j$, make $x_{ij}$ as large as possible in the order $i = 1$ to $m$ (top to bottom) noting $x_{ij} \leq k_{ij}$ until their total reaches purchase quantity $\tilde{D}_j$.

| Period $\rightarrow$ | $j = 1$ | 2 | $\ldots$ | n |
|---|---|---|---|---|
| Supplier $i = 1$ | $x_{11}$ $c_1$ | $x_{12}$ $c_1$ | $\ldots$ | $x_{1n}$ $c_1$ |
| 2 | $x_{21}$ $c_2$ | $x_{22}$ $c_2$ | $\ldots$ | $x_{2n}$ $c_2$ |
| $\vdots$ | | | | |
| $m$ | $x_{m1}$ $c_m$ | $x_{m2}$ $c_m$ | $\ldots$ | $x_{mn}$ $c_m$ |
| Quantity to be purchased | $D_1$ | $D_2$ | $\ldots$ | $D_n$ |

In each column of this array separately, start giving the variables the largest possible values subject to their upper bound, in the order top to bottom, until their total reaches the purchase quantity in that period. Let $\bar{x} = (\bar{x}_{ij})$ denote the solution obtained, the initial feasible solution. Let $\bar{y} = (\bar{y}_i)$ where $\bar{y}_i = \sum_{j=1}^{n} \bar{x}_{ij}$. Then

| | |
|---|---|
| $\sum_{i=1}^{m} c_i \bar{y}_i + \alpha \sum_{j=1}^{n} \tilde{s}_j$ | where $\tilde{s}_j$ comes from Step 0, is a lower bound for the optimum total cost. |
| $g(\bar{x}, \tilde{s}, \bar{y})$ | is an upper bound for the optimum total cost. |

If the difference between the two bounds is small, accept $(x^* = \bar{x}, \tilde{s}, y^* = \bar{y})$ as the approximate optimum solution with its interpretation given at the end of Step 2, and terminate. Otherwise go to Step 2.

**Step 2: Local search to move to a better solution**

Starting with $(\bar{x}, \bar{y})$ obtained at the end of Step 1, apply Step 2 as described in Section 3 until an $(x^*, y^*)$ that is a local optimum is obtained. Then accept $(x^*, y^*, \tilde{D}, \tilde{u}, \tilde{\eta}, \tilde{s})$ as an approximate optimum with the following interpretations.

| | |
|---|---|
| $x^* = (x^*_{ij})$ | gives amounts to purchase from various suppliers in various periods |
| $y^* = (y^*_i)$ | total amounts over all periods purchased from the various suppliers |
| $\tilde{D} = (\tilde{D}_j)$ | from Step 0, gives total quantities purchased from all suppliers in various periods |
| $\tilde{u} = (\tilde{u}_j)$ | from Step 0, gives the quantities to be sent to the buffer from storage in he various periods |
| $\tilde{\eta} = (\tilde{\eta}_j)$ | from Step 0, gives the quantities to be withdrawn from the buffer towards the demand in the various periods |
| $\tilde{s} = (\tilde{s}_j)$ | from Step 0, gives the unfulfilled demand in the various periods. |