

# Chapter 13

## Radioactive Decay

*Note to students and other readers: This Chapter is intended to supplement Chapter 6 of Krane's excellent book, "Introductory Nuclear Physics". Kindly read the relevant sections in Krane's book first. This reading is supplementary to that, and the subsection ordering will mirror that of Krane's, at least until further notice.*

### 13.1 The Radioactive Decay Law

#### Exponential decay law

Consider a system of particles,  $N_0$  in number at time,  $t = 0$ . Each of these particles has an independent, but equal probability of decay per unit time,  $\lambda$ . How many particles are observed at a later time? The traditional way of answering this question is to assume that  $N$  is large enough, that we may use calculus. Since particles are integral quantities, we recognize that this is, somewhat, a leap of faith!

Thus, the change in  $N$  is given by:

$$\begin{aligned}dN &= -\lambda N dt; & N(0) &= N_0 \\ \frac{dN}{N} &= -\lambda dt \\ d[\log N] &= -\lambda dt \\ \log N - \log N_0 &= -\lambda t \\ \log N &= \log N_0 - \lambda t \\ N &= N_0 \exp(-\lambda t)\end{aligned}\tag{13.1}$$

Thus we have derived the well-known *exponential decay law*,  $N(t) = N_0 e^{-\lambda t}$ .

**Half-life**

The *half-life*,  $t_{1/2}$ , is defined as follows:

$$\frac{N(t + t_{1/2})}{N(t)} \equiv \frac{1}{2} = \frac{N_0 \exp(-\lambda t - \lambda t_{1/2})}{N_0 \exp(-\lambda t)} = \exp(-\lambda t_{1/2}) ,$$

or,

$$t_{1/2} = \frac{\log 2}{\lambda} \approx \frac{0.693}{\lambda} . \quad (13.2)$$

Thus we see, a population of  $N$  radioactive particles at  $t$  would be reduced by half (on average) at time  $t + t_{1/2}$ .

**Lifetime, or mean lifetime**

The exponential law can also be interpreted as the *decay probability* for a single radioactive particle to decay in the interval  $dt$ , about  $t$ . This probability,  $p(t)$ , properly normalized, is given by:

$$p(t)dt = \lambda e^{-\lambda t} dt \quad ; \quad \int_0^{\infty} p(t)dt = 1 . \quad (13.3)$$

The we see that the probability a particle decays within time  $t$ ,  $P(t)$  is given by,

$$P(t) = \int_0^t p(t')dt' = 1 - e^{-\lambda t} . \quad (13.4)$$

The *mean lifetime* or *lifetime* of a particle,  $\tau$ , is evaluated by calculating  $\langle t \rangle$ , using the probability distribution (13.3):

$$\tau = \lambda \int_0^{\infty} t e^{-\lambda t} dt = \frac{1}{\lambda} . \quad (13.5)$$

**Activity**

The number of decays,  $\Delta N$ , observed from  $t$  and  $t + \Delta t$ , obtained from (13.1) is:

$$\Delta N = N(t) - N(t + \Delta t) = N_0 e^{-\lambda t} (1 - e^{-\lambda \Delta t}) .$$

If  $\Delta t \ll \tau$ , then, in the limit as  $\Delta t \rightarrow 0$ , we may rewrite the above as:

$$\left| \frac{dN}{dt} \right| = \lambda N_0 e^{-\lambda t} = \lambda N(t) \equiv \mathcal{A}(t) = \mathcal{A}_0 e^{-\lambda t}, \quad (13.6)$$

defining the *activity*,  $\mathcal{A}(t)$ , and its initial value,  $\mathcal{A}_0$ . Activity is usually what is measured, since  $N_0$  and  $N(t)$  are usually unknown, nor of particular interest in many applications. What is generally of real interest is the activity of a source, and, consequently, the ability of the radiation from a source, to interact.

It must be emphasized that (13.6) is an approximate relationship, based on  $\Delta t \ll \tau$ . Consequently, using (13.6) in any other expression, is subject to the same constraint.

The previous considerations only apply to the decay of a single isotope,  $N_1$ , to another (presumably stable) nucleus,  $N_2$ .

The differential equations describing the decline of  $N_1$  and the growth of  $N_2$  are given as follows:

$$\begin{aligned} dN_1 &= -N_1 \lambda dt \\ dN_2 &= N_1 \lambda dt \\ d(N_1 + N_2) &= 0, \end{aligned} \quad (13.7)$$

with solutions:

$$\begin{aligned} N_1 &= N_1(0) e^{-\lambda t} \\ N_2 &= N_2(0) + N_1(0)(1 - e^{-\lambda t}) \\ N_1 + N_2 &= N_1(0) + N_2(0) \end{aligned} \quad (13.8)$$

### One isotope, two decay channels

Now imagine that  $N$  can decay, with  $\lambda_a$  into  $N_a$ , or into  $N_b$  with  $\lambda_b$ . The total decay rate is  $\lambda_t = \lambda_a + \lambda_b$ .

The differential equations are:

$$\begin{aligned} dN &= -N \lambda_t dt \\ dN_a &= N \lambda_a dt \\ dN_b &= N \lambda_b dt \\ d(N + N_a + N_b) &= 0, \end{aligned} \quad (13.9)$$

with solutions:

$$\begin{aligned}
 N &= N(0)e^{-\lambda t} \\
 N_a &= N_a(0) + (\lambda_a/\lambda_t)N(0)(1 - e^{-\lambda t}) \\
 N_b &= N_b(0) + (\lambda_b/\lambda_t)N(0)(1 - e^{-\lambda t}) \\
 N + N_a + N_b &= N(0) + N_a(0) + N_b(0) .
 \end{aligned} \tag{13.10}$$

### One parent, many stable daughters

The results of the one parent (nuclear isotope  $\longrightarrow$  2 stable daughters), can easily be generalized to many daughters. The differential equations are:

$$\begin{aligned}
 dN &= -N\lambda_t dt \\
 dN_1 &= N\lambda_1 dt \\
 dN_2 &= N\lambda_2 dt \\
 &\vdots \\
 dN_n &= N\lambda_n dt \\
 d\left(N + \sum_{i=1}^n N_i\right) &= 0 ,
 \end{aligned} \tag{13.11}$$

where  $\lambda_t = \sum_{i=1}^n \lambda_i$ . The solutions are given by:

$$\begin{aligned}
 N &= N(0)e^{-\lambda t} \\
 N_i &= N_i(0) + (\lambda_i/\lambda_t)N_0(1 - e^{-\lambda t}) \\
 N + \sum_{i=1}^n N_i &= N(0) + \sum_{i=1}^n N_i(0) .
 \end{aligned} \tag{13.12}$$

The quantity  $\lambda_i/\lambda_t$  is called the *branching ratio* for decay into channel  $i$ .

### Two isotopes, independent decay channels

In this case, the total activity measured is:

$$\mathcal{A}(t) = \mathcal{A}_0^a e^{-\lambda_a t} + \mathcal{A}_0^b e^{-\lambda_b t} , \tag{13.13}$$

where  $a$  and  $b$  label the two different isotopes.

The question arises: How to determine  $\mathcal{A}_0^a$ ,  $\mathcal{A}_0^b$ ,  $\lambda_a$  and  $\lambda_b$ ? Krane outlines a strategy portrayed in his Figure 6.2. If, for example, isotope  $a$  has a considerably longer half-life, the linear tail of the log-linear plot of  $\mathcal{A}$  can be extrapolated backward to isolate  $\mathcal{A}_0^a$ . The slope of this line also yields  $\lambda_a$ . The one plots  $\log(\mathcal{A}(t) - \mathcal{A}_0^a e^{-\lambda_a t})$  that should give a straight line log-linear plot. The values of  $\mathcal{A}_0^b$  and  $\lambda_b$  can then be determined.

## 13.2 Quantum Theory of Radioactive Decay

The Quantum Theory of Radioactive Decay starts with a statement of Fermi's Golden Rule<sup>1</sup> #2, the equation from which decays rates, and cross sections are obtained. It is one of the central equations in Quantum Mechanics. Fermi's Golden Rule #2 for the transition rate (probability of transition per unit time),  $\lambda$ , is given by:

$$\lambda = \frac{2\pi}{\hbar} |\langle \psi_f | V_p | \psi_i \rangle|^2 \frac{dn_f}{dE_f}, \quad (13.14)$$

where  $\psi_i$  is the initial quantum state, operated on by a perturbation (transition) potential,  $V_p$ , resulting in the final quantum state,  $\psi_f$ . The factor  $dn_f/dE_f$  is called the *density of final states* [sometimes given the notation  $\rho(E_f)$ ]. The density of final states factor enumerates the number of possible final states (degeneracy) that can acquire the final energy  $E_f$ . It is not possible to express a less generic form of this factor, without a specific application in mind. It must be derived on a case-by-case basis, for a given application. We shall have opportunity to do this several times before the conclusion of this course.

### Application to nuclear $\gamma$ decay

In this case, we have the following situation:

$$\begin{aligned} N_i^* &\longrightarrow N_f + \gamma \\ \psi^{N_i^*} &\longrightarrow \psi^{N_f} \psi^\gamma \\ E_i &= E_f + E_\gamma + E_R. \end{aligned} \quad (13.15)$$

That is, a nucleus in an initial excited state,  $N_i^*$ , has a  $\gamma$ -transition to a final state  $N_f$ . The final state may be an intermediate (but lower in energy) excited state. (It is conventional to use an asterisk to represent an excited state.) Without loss of generality, we can assume, for

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<sup>1</sup>A derivation of Fermi's Golden Rule #2 is given at the end of this section.

the remaining discussion, that  $N_f$  represents the ground state. In (13.15) we see that the energy of the  $\gamma$  is given by the difference in energies of the two nuclear states, less the recoil energy,  $E_f$  imparted to the daughter (resultant) nucleus.

If the energy of the excited state is uncertain, we know that its lifetime and its energy are connected through the Heisenberg Uncertainty relationship:

$$\Delta E \Delta t \geq \hbar/2 .$$

This is a consequence of the wave description of matter. So,  $\Delta E \uparrow \Rightarrow \Delta t \downarrow$ , if the uncertainty in energy increases, the uncertainty in the lifetime decreases. Conversely,  $\Delta E \downarrow \Rightarrow \Delta t \uparrow$ , if the uncertainty in energy decreases, the uncertainty in the lifetime increases.

It remains to discover, therefore, what is the exact relationship between a state's lifetime, and the distribution of energies that are observed? The derivation is sketched below and the results presented. The details of the derivation are left to the optional section at the end of this section.

We start by assuming that the final nuclear state is given by:

$$\begin{aligned} \psi^{N_f}(\vec{x}, t) &= \psi^{N_f}(\vec{x}) e^{iE_f t/\hbar} e^{-t/(2\tau)} \\ |\psi^{N_f}(\vec{x}, t)|^2 &= |\psi^{N_f}(\vec{x})|^2 e^{-t/\tau} , \end{aligned} \quad (13.16)$$

to agree with our discussion, in the last section, of the probability of decay of a single particle. Recall that  $\tau$  is the ‘‘lifetime’’.

The derivation in the next section reveals that the probability of observing decay energy  $E$ ,  $p(E)$ , is given by:

$$p(E) = \frac{\Gamma}{2\pi} \frac{1}{(E - E_f)^2 + (\Gamma/2)^2} , \quad (13.17)$$

where  $\Gamma \equiv \hbar/\tau$ . This probability distribution is normalized:

$$\int_{-\infty}^{\infty} dE p(E) = 1 .$$

The peak of this distribution is  $p(E_f)$  and  $\Gamma$  width of the distribution at half-maximum. That is,  $p(E_f \pm \Gamma/2) = \frac{1}{2}p(E_f)$ . Also,  $\langle E \rangle = E_f$ .

This distribution is called the *Lorentz distribution*, or simply, the *Lorentzian function*. It is also known as *Cauchy-Lorentz distribution*, the *Cauchy distribution*, or the *Breit-Wigner*

*distribution.* Seems that everyone wants to crash this party! Wikipedia has a useful page on this topic.

The unnormalized Lorentzian is plotted in Figure 13.1, while the normalized Lorentzian is plotted in Figure 13.2.

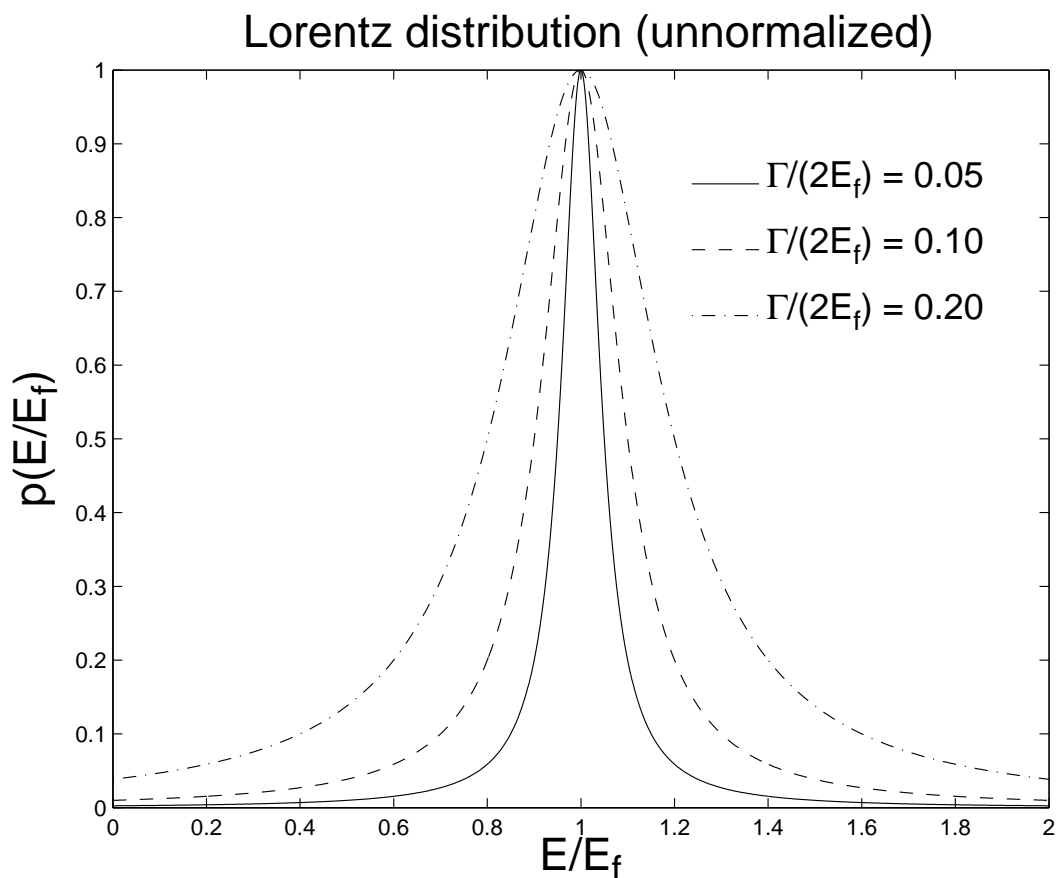


Figure 13.1: The unnormalized Lorentzian.

We note that this spread of energies is intrinsic; it has nothing to do with measurement uncertainties. Even with a perfect detector, we would observe this spread of detected energies.

Finally, we investigate, for typical  $\gamma$ -transition lifetimes, what is the expected range of energy-spreads that is likely to be observed? For  $10^{-12} \text{ s} < \tau < \infty$ , typical for  $\gamma$  decays, we find that  $0.00066 \text{ eV} > \Gamma > 0$ . That is,  $\gamma$  spectroscopy can exquisitely isolate the individual energy levels. The situation is drastically different for high-energy physics, where intrinsic widths can be of the order of 1 GeV or so. Nearby excitation can overlap with each other, and the identification of excited states (of hadrons) can be very difficult.

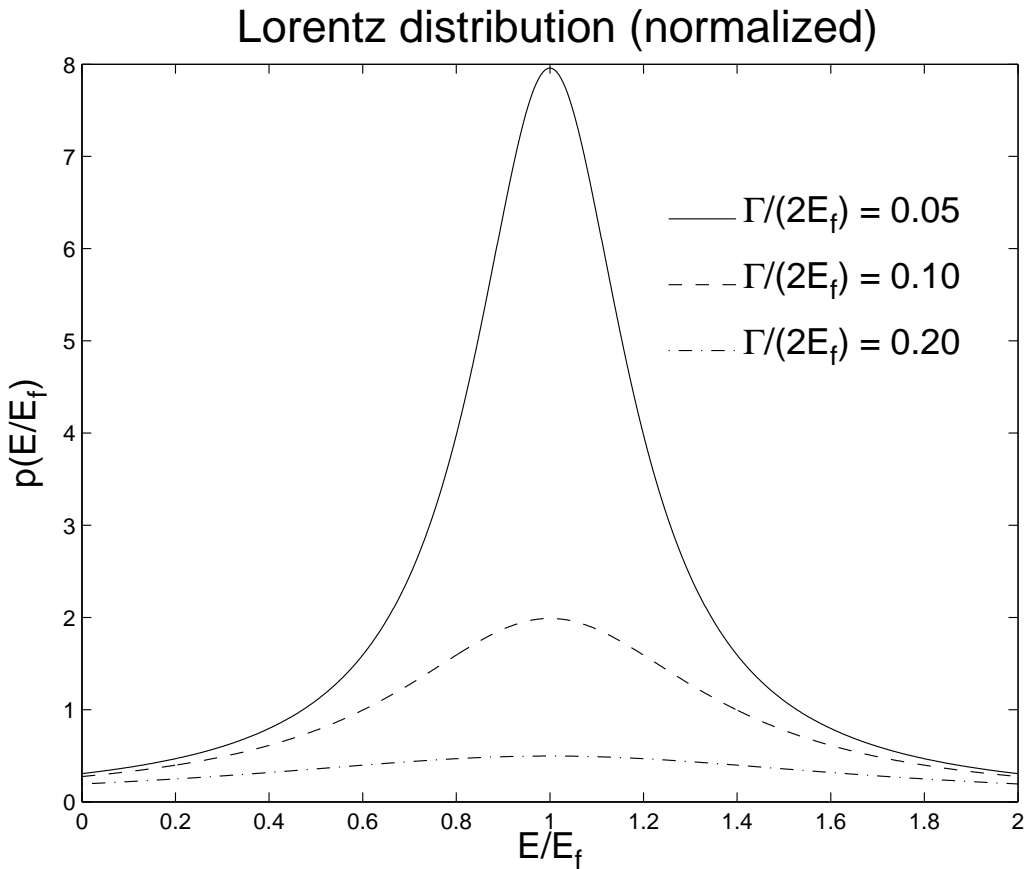


Figure 13.2: The normalized Lorentzian.

### Density of final states

Given a final energy of a quantum system, how many states,  $dn_f$  fall in the range,  $E_f \rightarrow E_f + dE_f$ ? If we can calculate this, then we can form the ratio  $dn_f/E_f$ , the *density of final states*. We proceed with the derivation, assuming that there is just one free particle in the final state. This will cover most situations of interest in this course.

We start by expressing the free particle wave function, as it would exist in a cubical box ( $V = 0$  inside,  $V \rightarrow \infty$  outside, one corner at the origin of the coordinate system)) with side  $L$ , and take the limit  $L \rightarrow \infty$  at the end. As we discovered in NERS311, the wavefunction is given by:

$$\psi_{n_x, n_y, n_z}(x, y, z) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right), \quad (13.18)$$

where  $1 \leq n_i < \infty$  are the 3 quantum numbers ( $i = 1, 2, 3$ ) in the 3D system. The 3



momentum components are given by:

$$p_i = \frac{n_i \pi \hbar}{L} . \quad (13.19)$$

We can imagine a 3D lattice of  $(n_x, n_y, n_z)$  points occupying the  $(+, +, +)$  octant in space (since the  $n_i$ 's are positive).  $L$  is very big, and so, we can treat the  $n_i$  as continuous. Let  $\vec{n} \equiv n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$  and  $n^2 = \vec{n} \cdot \vec{n}$ . The number of states,  $dn_f$  in a shell of thickness  $dn$  (all with the same momentum and energy) is given by:

$$\begin{aligned} dn_f &= \frac{1}{8} \frac{4\pi}{3} [(n + dn)^3 - n^3] \\ &= \frac{\pi}{6} [n^3 + 3n^2 dn + 3n(dn)^2 + (dn)^3 - n^3] \\ &= \frac{\pi}{6} [3n^2 dn + 3n(dn)^2 + (dn)^3] \\ &\rightarrow \frac{\pi}{2} n^2 dn . \end{aligned} \quad (13.20)$$

Thus

$$\frac{dn_f}{dE_f} = \frac{\pi}{2} n^2 \frac{dn}{dE_f} . \quad (13.21)$$

Now, it is a simple matter of relating  $E_f$  to the magnitude of the quantum number vector  $n$  and we are done!

### $\gamma$ -decay

For one photon in the final state, from (13.19)

$$E_\gamma = cp_\gamma = n\pi\hbar c/L . \quad (13.22)$$

Thus,

$$\begin{aligned} n &= \frac{E_\gamma L}{\pi\hbar c} \\ \frac{dn}{dE_\gamma} &= \frac{L}{\pi\hbar c} . \end{aligned} \quad (13.23)$$

Recognizing that  $E_f = E_\gamma$ , the results of (13.23) used in (13.21) gives:

$$\frac{dn_f}{dE_\gamma} = \frac{1}{2\pi^2} \frac{L^3}{(\hbar c)^3} E_\gamma^2. \quad (13.24)$$

Specializing to  $\gamma$  decay, we adapt (13.14) using the notation of (13.15) along with result of (13.24):

$$\lambda_\gamma = \frac{1}{\pi\hbar(\hbar c)^3} |\langle \psi^{N_f} [L^{3/2}\psi^\gamma] | V_p | \psi^{N_i} \rangle|^2 E_\gamma^2. \quad (13.25)$$

Note that the  $L^{3/2}$  denormalizes the photon wavefunction. Let us call this denormalized wavefunction  $L^{3/2}\psi^\gamma \equiv \tilde{\psi}^\gamma$ . It has no units associated with it. Hence the expression,  $|\langle \psi^{N_f} \tilde{\psi}^\gamma | V_p | \psi^{N_i} \rangle|^2$  has units  $(\text{energy})^2(\text{length})^3$ , and has a scale proportional to the volume of the nucleus. Hence,  $\lambda_\gamma$  has units  $\text{s}^{-1}$ , and is correctly dimensioned.

**Derivation of Fermi's Golden Rule #2**

Here is the theoretical background for the earlier parts of this section. It's optional material, but may be of interest to some readers.

Fermi's Golden Rule #2 is one of the central equations in radiation physics, as it is employed to obtain decay rates and cross sections. Thus, a clear derivation is called for.

Consider the Schrödinger equation for a single particle in a static binding potential:

$$\mathcal{H}_0\Psi(\vec{x}, t) = \frac{i}{\hbar}\Psi(\vec{x}, t) \quad (13.26)$$

where

$$\begin{aligned} \mathcal{H}_0 &= \mathcal{T} + V(\vec{x}) \\ \mathcal{T} &= -\frac{\hbar^2}{2m}\nabla^2 \end{aligned} \quad (13.27)$$

We know that such a potential has a set of orthonormal eigenstates:

$$|j\rangle = \Psi_j(\vec{x}, t) = \Psi_j(\vec{x})e^{-E_j t/\hbar}, \quad (13.28)$$

with eigenenergies,  $E_j$ . These eigenstates are orthonormal, that is,

$$\langle i|j\rangle = \delta_{ij}. \quad (13.29)$$

We know that the  $E_j$ 's are constants, and fixed. By Heisenberg's Uncertainty Principle, we also know that all eigenstates are stable, as there is no mechanism for decay. In Nature, we know that excited states eventually decay to the ground state, and the purpose of this derivation is to obtain an expression for that decay rate.

We start by assuming that there is a perturbation potential that is time dependent,  $V_p(\vec{x}, t)$ .

Now we solve:

$$(\mathcal{H}_0 + V_p)\Psi = \frac{i}{\hbar}\frac{\partial\Psi}{\partial t}, \quad (13.30)$$

where  $\Psi$  is the general solution to the entire problem, with both static and perturbation potentials included.

To start, we write  $\Psi(\vec{x}, t)$  as a superposition of the eigenstates of the  $\mathcal{H}_0$  operator, that is:

$$\Psi(\vec{x}, t) = \sum_j a_j(t) \Psi_j(\vec{x}, t) \quad (13.31)$$

Taking the partial derivative (13.31) with respect to  $t$  gives: (Henceforth, for brevity, obvious functional dependences on space and time will usually be suppressed.)

$$\frac{\partial \Psi}{\partial t} = \sum_j \left( \dot{a}_j - i \frac{E_j}{\hbar} a_j \right) \Psi_j . \quad (13.32)$$

(13.32) + (13.31)  $\longrightarrow$  (13.30)  $\Rightarrow$

$$\sum_j a_j (\mathcal{H}_0 - E_j) \Psi_j + \sum_j (a_j V_p - i \hbar \dot{a}_j) \Psi_j = 0 . \quad (13.33)$$

The first summation is zero, because each  $\Psi_j$  is a eigenfunction of the unperturbed  $\mathcal{H}_0$  with eigenenergy  $E_j$ . Thus,

$$\sum_j (a_j V_p - i \hbar \dot{a}_j) \Psi_j = 0 , \quad (13.34)$$

or,

$$\sum_j (a_j V_p - i \hbar \dot{a}_j) |j\rangle = 0 . \quad (13.35)$$

Let  $|f\rangle$  be the state that the excited states  $|j\rangle$  transitions to. You can think of  $|f\rangle$  as the ground state, or at least a lower excited state.

$\langle f| \otimes$  (13.35)  $\Rightarrow$

$$\sum_j (a_j \langle f| V_p |j\rangle e^{i(E_j - E_f)/\hbar} - i \hbar \dot{a}_j) \delta_{jf} = 0 . \quad (13.36)$$

Using the shorthand notation  $V_{jf} \equiv \langle f| V_p |j\rangle$  and  $\omega_{jf} \equiv (E_j - E_f)/\hbar$ , we have:

$$i \hbar \dot{a}_f = \sum_j a_j V_{jf} e^{i(E_j - E_f)/\hbar} . \quad (13.37)$$

(13.37) represents, at least in principle, an exact solution the problem. All one needs to do is to set an initial condition, say,  $a_n(0) = 1$  (the excited state) and then al the other  $a$ 's, potentially an infinite number (!) to zero, and then let the solution evolve. Note that every single eigenstate can be involved in the eventual de-excitation of  $|n\rangle$ . This approach is more amenable to numerical solution. So, to proceed with the analysis, we make the ...

**Small perturbation approximation**

In this approximation, we only have two states, the initial excited and final states,  $|i\rangle$  and  $|f\rangle$ . None of the other states are assumed to be involved. In the spirit of this approximation, we treat the  $a$ 's on the right hand side of (13.37) as constants. (This is how the system would evolve for small  $t$  for any perturbation, large or small.)

Hence we set  $a_i(t) = 1 \forall t$ ,  $a_f(0) = 0$ , we allow these to change with time, and all the other  $a$ 's are set to zero for all time. This allows us to integrate the equation, resulting in:

$$a_f = V_{if} \frac{1 - e^{i\omega_{if}t}}{\hbar\omega_{if}} . \quad (13.38)$$

Now, we evaluate the occupation probability of the state to which the transition is made,

$$P = |a_f|^2 = |V_{if}|^2 \frac{(1 - e^{i\omega_{if}t})(1 - e^{-i\omega_{if}t})}{(\hbar\omega_{if})^2} . \quad (13.39)$$

Using some trigonometric identities, this can be recast into the following form:

$$P = |a_f|^2 = \frac{|V_{if}|^2 \sin^2(\omega_{if}t/2)}{\hbar^2 (\omega_{if}/2)^2} . \quad (13.40)$$

The derivation of (13.40) assumed that the energy of the initial state  $|i\rangle$  is precisely known. However, we know from the Heisenberg's Uncertainty Principle,  $\Delta E \Delta t \geq \hbar/2$ , that the energy of an excited state can not be known precisely, but distributed in some way. So, assume that the energy of the excited state is distributed according to some distribution  $\rho(\omega)$ . We must integrate over all of these to obtain the occupation probability:

$$P = \int_{-\infty}^{\infty} d\omega \frac{|V_{if}|^2}{\hbar^2} \rho(\omega) \frac{\sin^2((\omega_{if} - \omega)t/2)}{((\omega_{if} - \omega)/2)^2} . \quad (13.41)$$

The

$$\frac{\sin^2((\omega_{if} - \omega)t/2)}{((\omega_{if} - \omega)/2)^2}$$

term in the above equation acts as a delta function for large  $t$ , narrowing as  $t$  increases. We eventually want to consider the decay of the excited state to the final state, so we take the large  $t$  limit to obtain, after a change of variables:

$$P = \frac{|V_{if}|^2}{\hbar^2} \rho(\omega_{if}) 2t \int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} . \quad (13.42)$$

The integral evaluates numerically to  $\pi$ , thus

$$P = \frac{2\pi}{\hbar^2} |V_{if}|^2 \rho(\omega_{if}) t \quad (13.43)$$

We can also rewrite  $\rho(\omega_{if})$  in terms of  $E_{if}$ . Since  $E_{if} = \hbar\omega_{if}$ ,

$$P = \frac{2\pi}{\hbar} |V_{if}|^2 \rho(E_{if}) t . \quad (13.44)$$

Finally, the rate of decay,  $\lambda = dP/dt$ . Hence,

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$$\lambda = \frac{2\pi}{\hbar} |V_{if}|^2 \rho(E_{if}) , \quad (13.45)$$


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and we have derived Fermi's Golden Rule #2.

A few comments are in order.

The “blurring” function  $\rho(E_{if})$  is sometimes referred to as the “density of final states”. We had to introduce it, in a somewhat ad hoc fashion to recognize that excited states are, indeed, “blurred”. However, it is fascinating to note, that this blurring is directly connected to the existence of final states for the system to accept the decay. For example, a typical nuclear decay involves the release of a  $\gamma$ . Unless this  $\gamma$  has a quantum state to occupy it, there can be no quantum mechanical transition. Hence, our interpretation of the “blurring” of excited states depends on our ability to measure its decay. If there is no decay mode, then this density of states function drops to zero, the decay does not occur, and hence, the energy of the excited state is precise! (But not measurable!)

What is the nature of this “blurring”?

### **The Lorentz distribution**

If an excited state can decay, we may write its wavefunction in the following form:

$$\Psi(\vec{x}, t) = \Psi(\vec{x}) e^{iE_{if}t/\hbar} \frac{e^{-t/(2\tau)}}{\sqrt{\tau}} , \quad (13.46)$$

where  $\tau$  is its mean life. This interpretation follows directly from the probability density of the excited state:

$$|\Psi(\vec{x}, t)|^2 = |\Psi(\vec{x})|^2 \frac{e^{-t/\tau}}{\tau}, \quad (13.47)$$

giving the well-known exponential decay law, properly normalized over the domain  $0 \leq t < \infty$ . Here we are adopting the normalization convention that

$$\int_0^\infty dt |\Psi(\vec{x}, t)|^2 = |\Psi(\vec{x})|^2.$$

Just as the dynamic variables  $k$  and  $x$  are related by Fourier transforms in the operational sense, this is true as well for  $\omega$  and  $t$ . Hence the above distribution in time, namely  $e^{-t/\tau}$ , is converted to a distribution in frequency by its Fourier transform, namely,

$$\Psi_i(\vec{x}, \omega) = \Psi_i(\vec{x}) \frac{1}{\sqrt{2\pi\tau}} \int_0^\infty dt e^{i(\omega_i - \omega)t} e^{-t/(2\tau)}, \quad (13.48)$$

where  $\omega_i = E_i$  and  $\omega = E$ .

After performing the integral

$$\Psi_i(\vec{x}, \omega) = \Psi_i(\vec{x}) \frac{1}{\sqrt{2\pi\tau}} \frac{1}{i(\omega_i - \omega) + 1/(2\tau)}, \quad (13.49)$$

Therefore,

$$|\Psi_i(\vec{x}, \omega)|^2 = |\Psi_i(\vec{x})|^2 \frac{1}{2\pi\tau} \left( \frac{1}{(\omega_i - \omega)^2 + (1/(2\tau))^2} \right). \quad (13.50)$$

In terms of  $E$  rather than  $\omega$ ,

$$|\Psi_i(\vec{x}, E)|^2 = |\Psi_i(\vec{x})|^2 \frac{\Gamma}{2\pi} \left( \frac{1}{(E_i - E)^2 + (\Gamma/2)^2} \right), \quad (13.51)$$

where  $\Gamma \equiv \hbar/\tau$ .

Thus we have found the form of the Lorentz distribution:

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$$|\Psi_i(\vec{x}, E)|^2 = |\Psi_i(\vec{x})|^2 \frac{\Gamma}{2\pi} \left( \frac{1}{(E_i - E)^2 + (\Gamma/2)^2} \right). \quad (13.52)$$


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One may easily verify that:

$$\int_{-\infty}^{\infty} dE |\Psi_i(\vec{x}, E)|^2 = |\Psi(\vec{x})|^2 . \quad (13.53)$$

## 13.3 Production and Decay of Radioactivity

### Secular equilibrium

Consider a beam of radiation, with intensity  $I$ , that impinges upon a block of material containing nuclei of type “0”. There are  $N_0$  nuclei in the target to start, and the external radiation activates the material, producing nuclear species “1”. We make two assumptions:

1. The target is “thin enough”, so that the external radiation is not attenuated.
2. The irradiation is weak enough so that  $N_0$  does not decline.

With these assumptions,  $R$ , the rate of production of  $N_1$  is given by:

$$R = I\sigma N_0 , \quad (13.54)$$

where  $\sigma$  is the radioactivity production cross section.

The differential equation describing the production of  $N_1$  is given by:

$$dN_1 = Rdt - \lambda_1 N_1 dt , \quad (13.55)$$

where  $\lambda_1$  is the decay rate of nucleus “1”. Solving (13.55):

$$\begin{aligned} \dot{N}_1 + \lambda_1 N_1 &= R \\ (\dot{N}_1 + \lambda_1 N_1)e^{\lambda_1 t} &= Re^{\lambda_1 t} \\ \frac{d}{dt}(N_1 e^{\lambda_1 t}) &= Re^{\lambda_1 t} \\ N_1 e^{\lambda_1 t} - N_1(0) &= \frac{R}{\lambda_1}(e^{\lambda_1 t} - 1) \\ N_1 - N_1(0)e^{-\lambda_1 t} &= \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) \\ N_1 &= N_1(0)e^{-\lambda_1 t} + \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) . \end{aligned} \quad (13.56)$$



Usually the target contains no activity at  $t = 0$ . If  $N_1(0) = 0$ :

$$\begin{aligned} N_1 &= \frac{R}{\lambda_1}(1 - e^{-\lambda_1 t}) \\ \mathcal{A}_1 &= R(1 - e^{-\lambda_1 t}) . \end{aligned} \tag{13.57}$$

Condition	Beam	$\mathcal{A}_1(t)$	Description
$\lambda_1 t \ll 1$ i.e. $t \ll \tau$	ON	$R\lambda_1 t$	initial production
$\lambda_1 t \gg 1$ i.e. $t \gg \tau$	ON	$R$	secular equilibrium
$t \geq t_0$ , $t_0$ is arbitrary	OFF (at $t = t_0$ )	$N_1(t_0)\lambda_1 e^{-\lambda_1(t-t_0)}$	decay of activity
$t \geq t_0 \gg \tau$	OFF (at $t = t_0$ )	$R e^{-\lambda_1(t-t_0)}$	decay of activity (from secular equilibrium)

During the condition of *secular equilibrium*, the rate of production is the same as the rate of decay, producing an unchanging number of radioactive daughter nuclei. (There will be some statistic fluctuation of this number.)

A depiction of a nuclide reaching secular equilibrium is shown in Figure 13.3.

### A real-life engineering application

A typical engineering challenge, in the area of creating radioactive sources, is to minimize the cost of producing a given amount of activity. We model this as follows:

The cost,  $C$ , per unit activity, factoring start-up costs,  $S_0$ , (manufacture of the inactive source, delivery costs, operator start-up and take-down), and the cost of running the accelerator or reactor, per unit meanlife,  $R_0$ , is given as follows:

$$C = \frac{S_0 + R_0 x_0}{1 - e^{-x_0}} , \tag{13.58}$$

where  $x_0$  is the number of meanlives that the target is irradiated.

The optimization condition is given by:

$$S_0/R_0 = e^{x_0} - (1 + x_0) . \tag{13.59}$$

For small  $S_0/R_0$ , the optimum  $x_0 \approx \sqrt{2S_0/R_0}$ . For large  $S_0/R_0$ , the optimum  $x_0 \approx \log(S_0/R_0)$ .

The solutions for the optimum value of  $x_0$  can be obtained from Figure 13.4.

The figure has been split into two parts,  $x_0 < 1$  for which  $S_0/R_0$  is shown, and  $x_0 > 1$  for which  $\log(S_0/R_0)$  is shown. The approximations discussed above, are plotted as dotted lines.

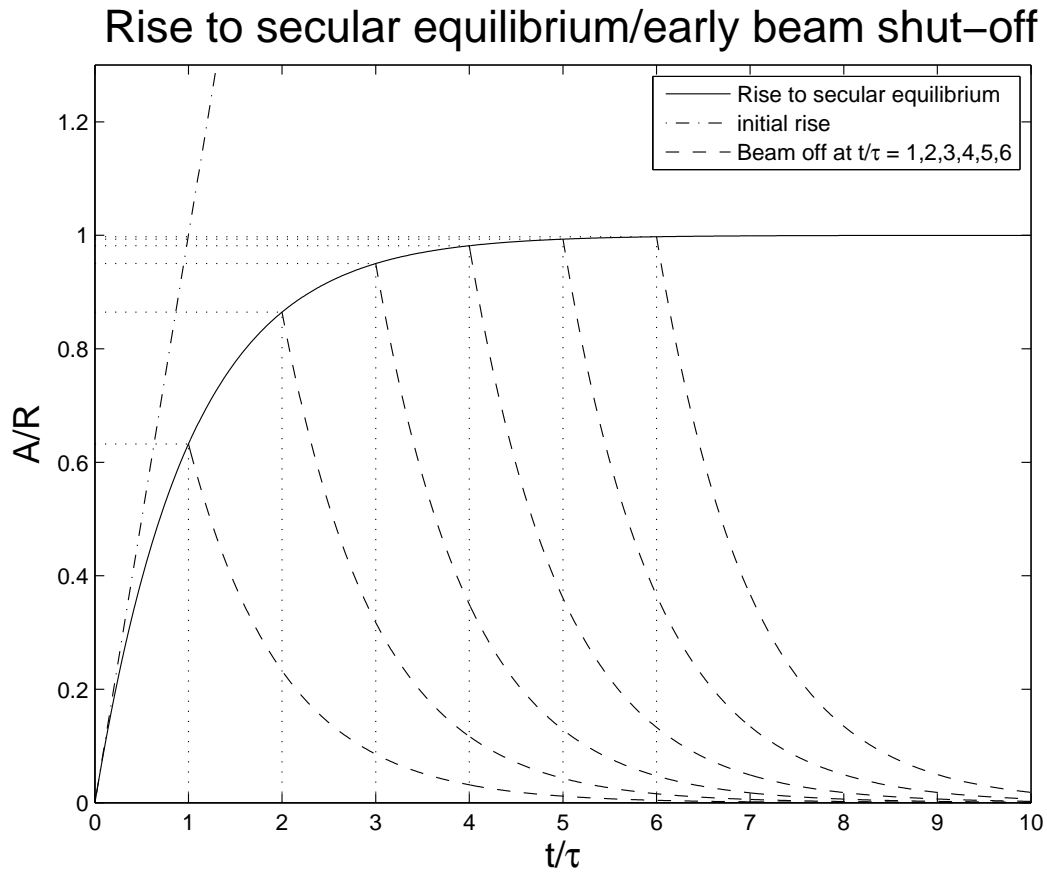


Figure 13.3: A nuclide reaching secular equilibrium is depicted. Also depicted are the decays of the source once the beam is shot off at  $t/\tau = 1, 2, 3, 4, 5$ .

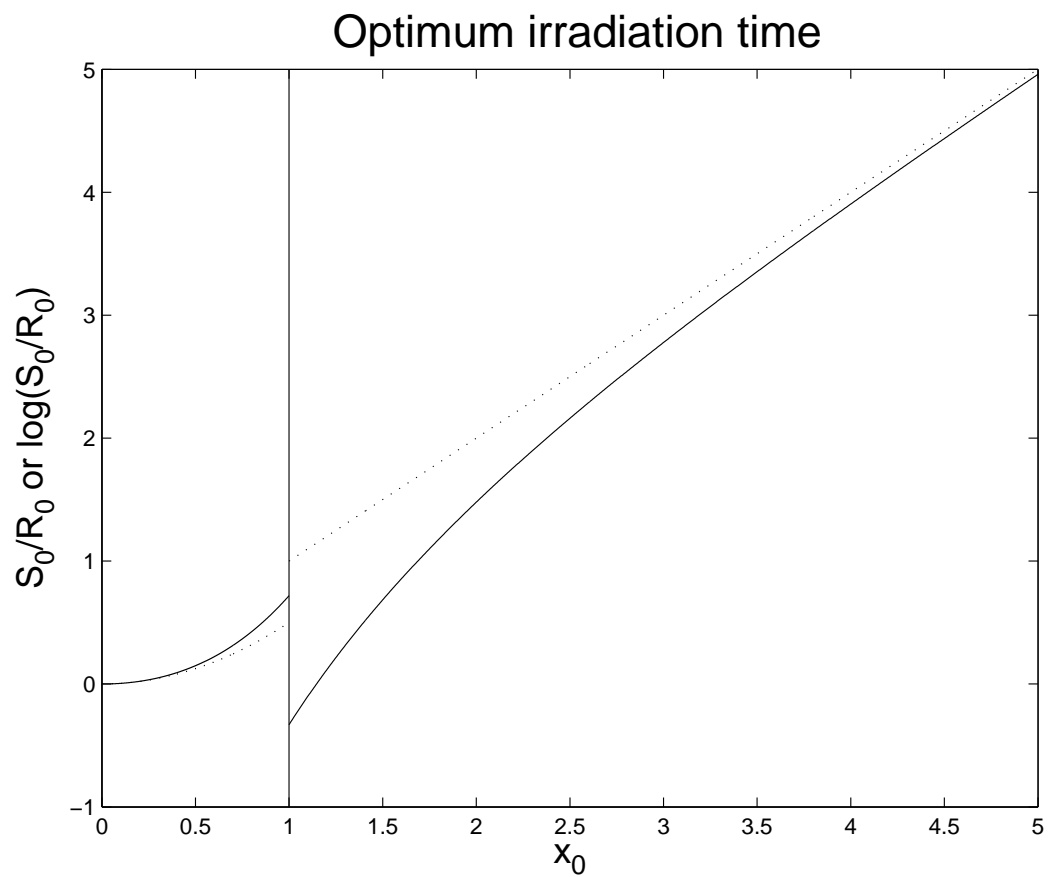


Figure 13.4: Optimization curves for determining  $x_0$ , as a function of  $\alpha = S_0/R_0$ .

## 13.4 Growth of Daughter Activities

Parent  $\Rightarrow$  Daughter  $\Rightarrow$  Granddaughter (stable)

We now describe the simplest decay *chain*, whereby a parent decays to an unstable daughter, that decay to a stable granddaughter. We'll use Krane's notation:

Differential equation	At $t = 0$	Description
$\dot{N}_1 = -\lambda_1 N_1$	$N_1(0) = N_0$	$N_1$ : Parent—rate constant $\lambda_1$ (decay only)
$\dot{N}_2 = \lambda_1 N_1 - \lambda_2 N_2$	$N_2(0) = 0$	$N_2$ : Daughter—growth, decay rate constant $\lambda_2$
$\dot{N}_3 = \lambda_2 N_2$	$N_3(0) = 0$	$N_3$ : Granddaughter—growth only
$\dot{N}_1 + \dot{N}_2 + \dot{N}_3 = 0$		“Conservation of particles”

The integrals are elementary, giving, for the  $N_i$ 's:

$$\begin{aligned}
 N_1 &= N_0 e^{-\lambda_1 t} \\
 N_2 &= N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \\
 N_3 &= N_0 - N_1 - N_2
 \end{aligned} \tag{13.60}$$

In this discussion, we concern ourselves mostly with the activity of the daughter, in relation to the parent, namely:

$$\mathcal{A}_2 = \mathcal{A}_1 \frac{\lambda_2}{\lambda_2 - \lambda_1} (1 - e^{-(\lambda_2 - \lambda_1)t}) , \tag{13.61}$$

and consider some special cases.

### Very long-lived parent: $\lambda_1 \lll \lambda_2$

In this case, the parent's meanlife is considered to be much longer than that of the daughter, essentially infinite within the time span of any measurement of interest. In other words, the activity of the parent is constant. In this case, (13.61) becomes:

$$\mathcal{A}_2 = \mathcal{A}_1(0)(1 - e^{-\lambda_2 t}) . \tag{13.62}$$

Comparing with (13.57), we see that (13.62) describes  $\mathcal{A}_2$ 's rise to secular equilibrium, with effective rate constant,  $\mathcal{A}_1(0)$ .

Long-lived parent:  $\lambda_1 \ll \lambda_2$ 

In this case, the parent lives much longer than the daughter, but the parent does have a measurable decline, within the time span of the measurement.

We rewrite (13.61) slightly as:

$$\mathcal{A}_2(t) = \mathcal{A}_1(t) \left( \frac{\lambda_2}{\lambda_2 - \lambda_1} (1 - e^{-(\lambda_2 - \lambda_1)t}) \right). \quad (13.63)$$

(13.63) describes  $\mathcal{A}_2(t)$  a modulation of  $\mathcal{A}_1(t)$ . An example is shown in figure 13.5.

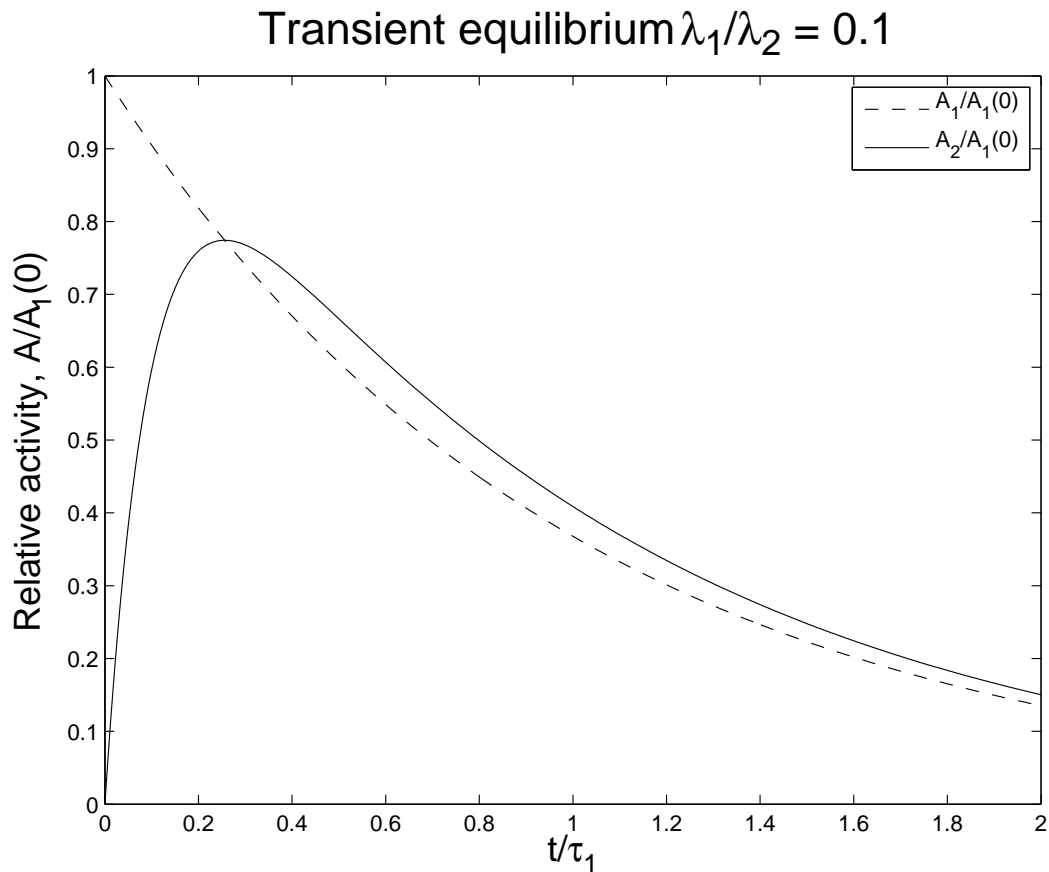


Figure 13.5: The relative activity of the daughter,  $\mathcal{A}_2$  and the parent,  $\mathcal{A}_1$ , for  $\lambda_2 = 10\lambda_1$ .

We see, from the figure, that the activity of the daughter,  $\mathcal{A}_2$ , rises quickly to match the parent, and then follows the parent activity closely. This latter condition, beyond a few meanlives of the daughter, is called the region of *transient secular equilibrium*, or more commonly, *transient equilibrium*. In the region of transient equilibrium, the activity of the

daughter is always slightly greater than that of the parent, with a temporal offset of about  $\tau_2$ .

**Special case:**  $\lambda = \lambda_1 = \lambda_2$

In this case, parent and daughter decay with the same meanlife. This is more of a mathematical curiosity, but should it occur, the result can be derived from (13.63) by assuming  $\lambda_1 = \lambda$ ,  $\lambda_2 = \lambda + \epsilon$ , performing a series expansion in  $\epsilon$ , and taking the  $\epsilon \rightarrow 0$  at the end. The result is:

$$\mathcal{A}_2(t) = \mathcal{A}_1(t)\lambda t . \quad (13.64)$$

### Series of Decays

In this case, we consider a series of descendents,  $N_1 \Rightarrow N_2 \Rightarrow N_2 \cdots \Rightarrow N_n$ , with rate constants,  $\lambda_1, \lambda_2 \cdots \lambda_n$ . That is, there are  $n$  generations, starting with the parent,  $N_1$ , and ending with a final stable (grand) <sup>$n-2$</sup> daughter, for  $n \geq 2$ .

The result of the solution of the differential equations:

Differential equation	At $t = 0$	Description
$\dot{N}_1 = -\lambda_1 N_1$	$N_1(0) = N_0$	$N_1$ : Parent—rate constant $\lambda_1$ (decay only)
$\dot{N}_i = \lambda_{i-1} N_{i-1} - \lambda_i N_i$	$N_i(0) = 0$	$N_i$ : (grand) <sup><math>n-2</math></sup> , decay rate constant $\lambda_i$
$\dot{N}_n = \lambda_n N_n$	$N_n(0) = 0$	$N_n$ : Stable end of chain (growth only)
$\sum_{i=1}^n \dot{N}_i = 0$		“Conservation of particles”

The result is given by the *Bateman equations*:

$$\begin{aligned} N_1 &= N_0 e^{-\lambda_1 t} \\ N_{1 < i < n} &= N_0 \sum_{j=1}^i \left( \frac{\prod_{k=1; k \neq i}^i \lambda_k}{\prod_{k=1; k \neq j}^i (\lambda_k - \lambda_j)} \right) e^{-\lambda_j t} \\ N_n &= N_0 - \sum_{i=2}^{n-1} N_i . \end{aligned} \quad (13.65)$$

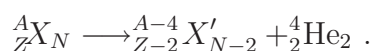
Note the exclusions in the product terms.

## 13.5 Types of Decays

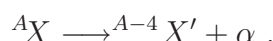
Details are covered elsewhere in this course. Here we just give a list.

### $\alpha$ decay

This is the form employed when an accurate mass calculation is to be performed. Electron masses cancel, and the small differences in electron binding energy are ignored.



Usually the following shorthand is employed:

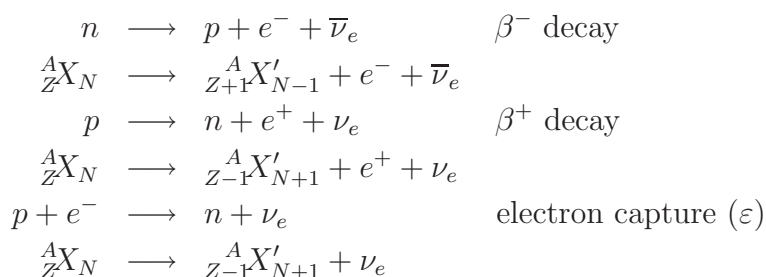


To come, more details on ...

Applications of  $\alpha$ -emitters:

- “Gadget” initiators ( $\alpha$ -decay followed by  $n$ -emission.)
- Search for superheavy elements. See <http://t2.lanl.gov/tour/shn.html>
- Smoke detectors.
- Power generation in space probes and artificial hearts.
- Unsealed source radiotherapy.
- Can be used to reduce static cling. (Really! I’m not kidding! The real question is “How?”)

### $\beta$ decay



A free neutron will  $\beta^-$  decay with a meanlife of 886.7(8) s. A neutron in a nucleus will  $\beta^-$  decay, but only when that process is favorable energetically.

Free protons do not decay, that is, it has never been observed. Proton decay is predicted by Grand Unified Theories (GUTs). However, the predicted probability of decay is exceedingly small. A lower bound for proton decay has been established experimentally, setting the half-life at greater than  $6.6 \times 10^{33}$  years. This is interesting, but of little consequence for Nuclear Engineering. Protons in a nucleus, if favored energetically, do  $\beta^+$  decay.

Finally, there is a process called *electron capture*, ( $\epsilon$ ), or *K-capture*, whereby a proton in a nucleus captures an orbital electron (usually from a 1s atomic orbital, and converts itself to a neutron.

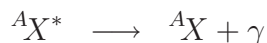
All these process result in a *electron neutrino*,  $\nu_e$ , or an *electron antineutrino*,  $\bar{\nu}_e$ . By convention, antiparticles, like the antiproton,  $\bar{p}$ , and the antineutron,  $\bar{n}$ , are written with an overline  $\bar{\phantom{x}}$  or an “overtilde”  $\tilde{\phantom{x}}$ . There are exceptions to this rule. The positron,  $e^+$  is the  $e^-$ 's antiparticle. However, it is never written as  $e^-$ .

To come, more details on ...

Applications of  $\beta$ -emitters:

- Betavoltaics (non-thermal) (long-life, low-power batteries).
- Radiotherapy (brachytherapy).
- PET (Positron Emission Tomography.)
- Radiopharmaceuticals.
- Quality assurance in large-scale paper production.
- Irradiation of domestic ruminant (cattle, goats, sheep, bison, deer, camels, alpacas, llamas) behinds to cure the effects of “fly strike”. (I’m not kidding about this one either.) (And I’d really rather not delve into the details of “fly strike”. Kindly google this one on your own.)

### $\gamma$ decay



Here, a nucleus in an excited state, denoted by the asterisk, decays via the  $\gamma$  process, to a lower excited state, or the ground state. All nuclei that are observed to have excited states, ( $A > 5$ ), have  $\gamma$  transitions.

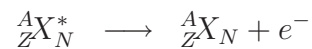


To come, more details on ...

Applications of  $\gamma$ -emitters:

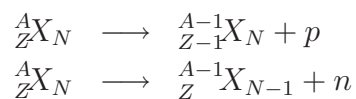
- Basic physics: Nuclear structure, astrophysics.
- Radiotherapy,  $^{60}\text{Co}$ ,  $^{137}\text{Cs}$ , brachytherapy..
- Sterilization of pharmaceutical products, food.
- Imaging vehicles for National Security Administration purposes.
- Industrial quality assurance.
- Discovery of oil. (Oil-well logging.)

### Internal conversion



Here, a nucleus in an excited state, de-excites by exchanging a virtual photon with a  $K$ -shell electron in a “close encounter” with the nucleus. The electron acquires the de-excitation energy and exits the nucleus.

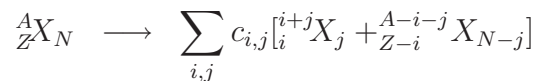
### Nucleon emission



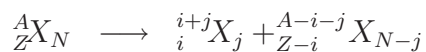
To come, more details on ...

Applications of nucleon emitters:

- Fission.

Spontaneous fission

Here a nucleus fractures into 2 (equation shown) or more (equation not shown) nuclei. This is similar to “normal” fission, except that it occurs spontaneously. Generally a spectrum of nuclei result, with the probabilities given by the  $c$  coefficients.

Cluster decay

where  $i, j > 2$ .

From:

[http://en.wikipedia.org/wiki/Cluster\\_decay!...](http://en.wikipedia.org/wiki/Cluster_decay!...)

*Cluster decay is a type of nuclear decay in which a radioactive atom emits a cluster of neutrons and protons heavier than an alpha particle. This type of decay happens only in nuclides which decay predominantly by alpha decay, and occurs only a small percentage of the time in all cases. Cluster decay is limited to heavy atoms which have enough nuclear energy to expel a portion of its nucleus.*

**13.6 Natural Radioactivity**

Not covered in 312

**13.7 Radioactive Dating**

Not covered in 312

## 13.8 Units for Measuring Radiation

Some useful radiometric quantities are listed in Table 13.1, along their traditional (outdated but still in some use), along with their new, almost universally adopted, SI<sup>2</sup>.

Quantity	Measures...	Old Units	New (SI) Units
Activity ( $\mathcal{A}$ )	decay rate	curie (Ci)	becquerel (Bq)
Exposure ( $X$ )	ionization in air	röntgen ( $R$ )	C/kg
Absorbed Dose ( $D$ )	Energy absorption	rad	gray (Gy = J/kg)
Dose Equivalent ( $DE$ )	Radiological effectiveness	rem	sievert (Sv = J/kg)

Table 13.1: Units for the measurement of radiation

SI units	Definition	Notes:
Bq	1 “decay”/s	derived
Gy	1 J/kg	derived
Sv	1 J/kg	derived
Traditional units	Conversion	Notes:
Ci	$3.7 \times 10^{10}$ Bq (exactly)	Charge/(1g) of dry, STP air
R	$1 \text{ esu}/(0.001293 \text{ g}) = 2.58 \times 10^{-4} \text{ C/kg}$	
rad	$1 \text{ ergs/g} = 10^{-2} \text{ Gy}$	
rem	$1 \text{ rem} = 10^{-2} \text{ Sv}$	

Table 13.2: Conversion factors

A derived unit is one that is based upon the seven base units in SI, namely: m, kg, s, A, K, mol (mole), cd (candela, luminosity).

### Activity

Activity ( $\mathcal{A}$ ) has been covered already. However, the units of measurement were not discussed. The traditional unit, the *curie*, (Ci), was named in honor of Marie Curie, and the modern unit in honor of Henri Bequerel.

<sup>2</sup>SI stands for *le Système International d’unités*, that was adopted universally (almost) in 1960. For more information, see: [http://en.wikipedia.org/wiki/SI\\_units](http://en.wikipedia.org/wiki/SI_units). SI is an abbreviation of a French-language phrase. France also houses the international metrology, the BIPM (*le Bureau International des Poids et Mesures*.) See, as well, <http://en.wikipedia.org/wiki/BIPM>.

**Exposure**

Exposure, given the symbol  $X$ , is defined as

$$X = \lim_{\Delta m \rightarrow 0} \frac{\Delta Q}{\Delta m}, \quad (13.66)$$

where  $\Delta Q$  is the amount of charge of one sign produced in, dry, STP air. (NIST definition: STP at STP: 20°C (293.15 K, 68°F, and an absolute pressure of 101.325 kPa (14.696 psi, 1 atm).) There are no accepted derived SI unit for exposure. Exposure measurements are probably the most accurately measured radiometric quantity.

**Absorbed Dose**

Absorbed dose measures the energy absorbed in matter, due to radiation. The traditional unit, the rad, has been supplanted by the Gy.

**Dose Equivalent**

The tradition unit, the rem (for röntgen equivalent man) and its modern counterpart, the Sv, attempts to accounts for the radiological damage from different “qualities” (species of particle imparting the dose) of radiation. Quality factors dependent on the energy of the radiation and are given in Table 13.2. The conversion from  $D$  to DE is given by:

$$\text{DE} = \sum_i \int dE D_i(E) \times \text{QF}_i(E), \quad (13.67)$$

where we sum over radiation types, and integrate over the energy of the radiation that imparts dose.

Radiation	Energy	QF
X-rays, $\gamma$ , $e^\pm$ , $\mu^\pm$	all	1
$p$ (non-recoil)	$> 1$ MeV	5
$n$	$< 10$ keV	5
	10–100 keV	10
	10 keV – 2 MeV	20
	2–20 MeV	10
	$> 20$ MeV	5
$\alpha$ , fission fragments, heavy nuclei	all	20

Table 13.3: Units for the measurement of radiation