

Eigenvalue Distributions of Kronecker Random Matrices

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1. Background and Introduction

Random matrix theory investigates the distribution of eigenvalues of random matrix ensembles. It has successfully been used as a model for applications in number theory and nuclear physics, among others.

Eigenvalue-Trace Lemma: Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) = \sum_{i=1}^N \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Normalization: We adjust the scale of all our eigenvalue distributions so that they have variance $\sigma^2 = 1$. By the Eigenvalue-Trace Lemma, $\sum_{i=1}^N \lambda_i(A)^2 = \text{Trace}(A^2) \sim N^2$, so to re-scale the eigenvalues we divide them by $\sqrt{N}$.

Eigenvalue Distribution: To each matrix A , assign a probability measure

$$\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^N \delta\left(x - \frac{\lambda_i(A)}{\sqrt{N}}\right).$$

Then

$$\begin{aligned} \int_a^b \mu_{A,N}(x) dx &= \frac{\#\{\lambda_i : \frac{\lambda_i(A)}{\sqrt{N}} \in [a, b]\}}{N}, \\ k^{\text{th}} \text{ moment} &= \int_{-\infty}^{\infty} x^k \mu_{A,N}(x) dx = \frac{\sum_{i=1}^N \lambda_i(A)^k}{N^{\frac{k}{2}+1}} \\ &= \frac{\text{Trace}(A^k)}{N^{\frac{k}{2}+1}} = \frac{\sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}}{N^{\frac{k}{2}+1}}. \end{aligned}$$

Wigner's Semicircle Law: In the ensemble of $N \times N$ real Wigner matrices, for almost all matrices A as N approaches infinity

$$\mu_{A,N} \rightarrow \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2} & \text{if } |x| \leq 2 \\ 0 & \text{if } |x| > 2 \end{cases}$$

where $\mu_{A,N}$ is the probability measure

$$\mu_{A,N} = \frac{1}{N} \sum_{i=1}^N \delta\left(x - \frac{\lambda_i}{\sqrt{N}}\right)$$

for $\{\lambda_i\}_{i=1}^N$ the eigenvalues of A .

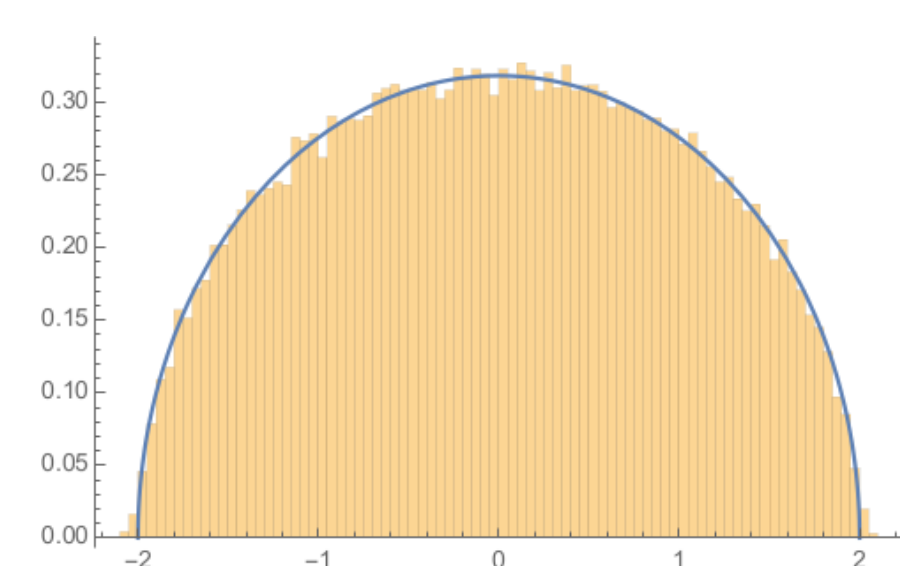


Figure 1.

Normalized eigenvalues from 500 100×100 real symmetric matrices.

2. Kronecker Products of Random Matrices

Definition: The Kronecker product of an $n \times n$ matrix A and an $m \times m$ matrix B is the $nm \times nm$ block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}.$$

The Kronecker product has the property that if

A has eigenvalues λ_i , $1 \leq i \leq n$ and

B has eigenvalues μ_j , $1 \leq j \leq m$, then

$A \otimes B$ has eigenvalues $\lambda_i \mu_j$, $1 \leq i \leq n$, $1 \leq j \leq m$.

This property implies the following theorem.

Theorem: Let A be chosen at random from some matrix ensemble and B be chosen at random from a possibly different ensemble. If the average moments of the eigenvalue distributions of A and B all exist, then the average k^{th} moment of $A \otimes B$ is the product of the average k^{th} moments of A and B .

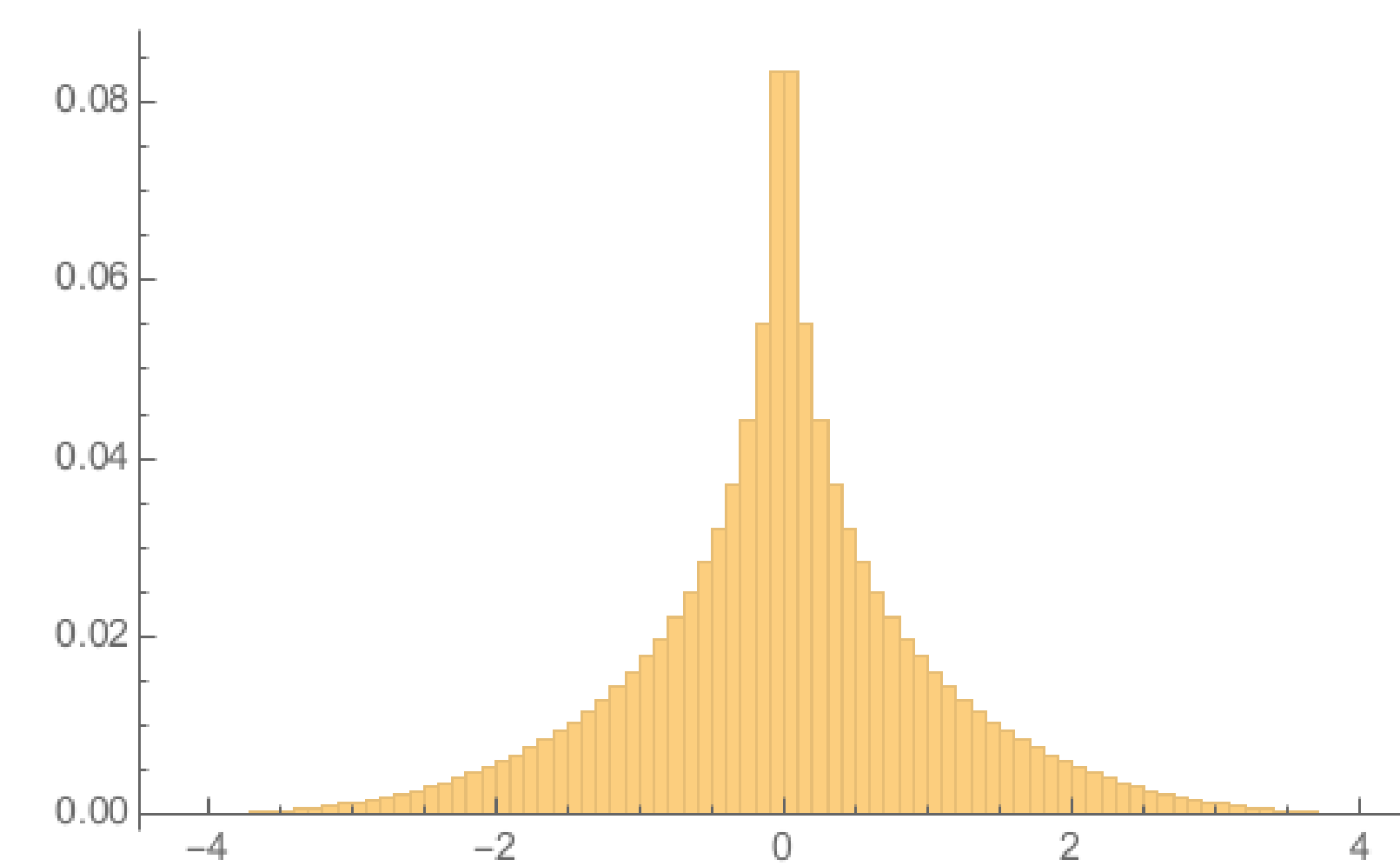


Figure 2.

Normalized eigenvalue distribution of the Kronecker product of two independent real symmetric matrices.

3. "Disco" Matrices

Definition: The disco matrix of two independent $n \times n$ matrices A and B is the $2n \times 2n$ block matrix

$$\text{Disco}(A, B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix} =: D.$$

We normalize the eigenvalues of $\text{Disco}(A, B)$, dividing by $2\sqrt{N}$. We explore the eigenvalue distributions of Disco matrices through the method of moments. We would like to compute the trace of powers of D in order to apply the Eigenvalue-Trace Lemma. Diagonalizing D gives

$$D^k = \begin{bmatrix} \mathbf{I}/2 & \mathbf{I}/2 \\ \mathbf{I}/2 & -\mathbf{I}/2 \end{bmatrix} \begin{bmatrix} (A+B)^k & 0 \\ 0 & (A-B)^k \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}.$$

So

$$\text{Trace}(D^k) = 2 \sum_{\substack{l=0 \\ l:\text{even}}}^k \sum_{i_1+\cdots+i_p=k-l} \sum_{j_1+\cdots+j_p=l} \text{Trace}(A^{i_1} B^{j_1} \cdots A^{i_p} B^{j_p}).$$

Note: The normalized eigenvalue distribution of $\text{Disco}(A, B)$ and that of $\text{Disco}(B, A)$ are the same.

Theorem: Suppose A and B are independent $n \times n$ matrices chosen from the same ensemble. The the normalized eigenvalue distribution of $\text{Disco}(A, B)$ is the same as that of A or B .

Conjecture: Suppose A and B are independent $n \times n$ matrices chosen from different ensembles. Then if the average k^{th} moments of A and B exist, the average k^{th} moment of $\text{Disco}(A, B)$ lies between them.

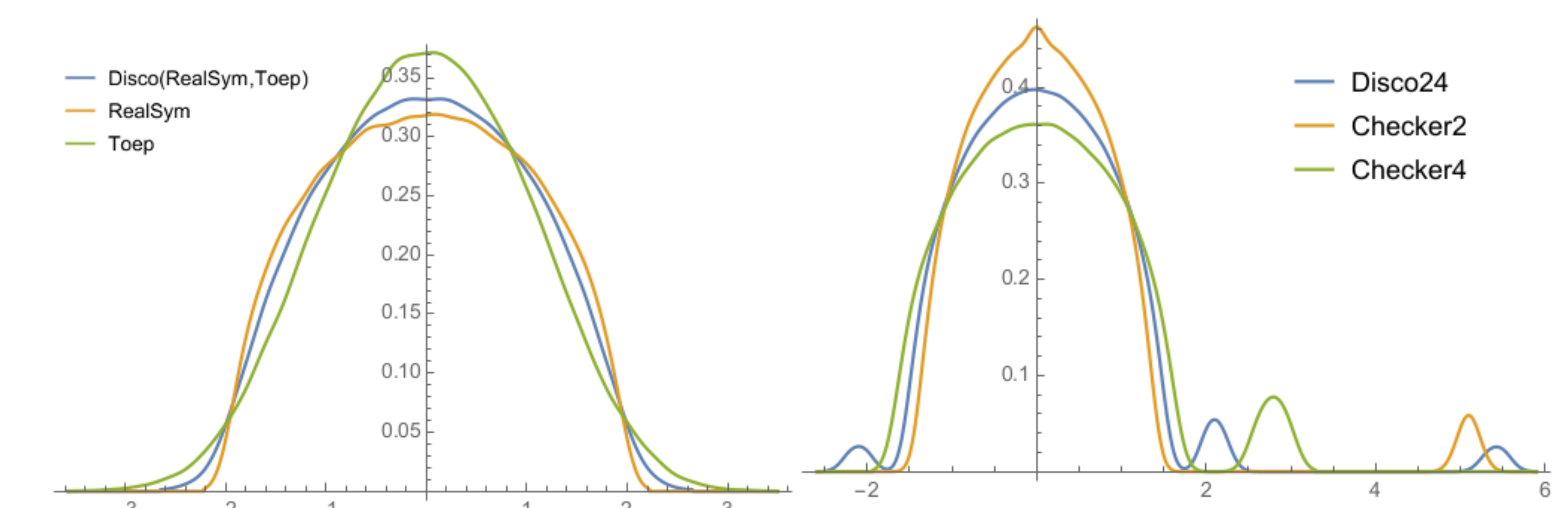


Figure 3. Real Symmetric, Toeplitz, and their Disco

Figure 4. Checkerboard with parameters 2 and 4, and their Disco

Remark: Although these statements tell us that the limiting distributions of eigenvalues should be well behaved under the operation $D(\cdot, \cdot)$, lower order terms have been observed, such as the "blip" in the distribution arising from the checkerboard ensemble. Numerical evidence shows that these effects are not as predictable under $\text{Disco}(\cdot, \cdot)$ (see Figure 4).

4. Acknowledgements

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