Graph-Restricted Decompositions: A Generalization of Zeckendorf's Theorem

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Outline

- Introduction to Zeckendorf Decompositions
- Framework: Graph-Restricted Decompositions
- Results on Decomposition Behavior
- Distributions of Number of Summands
- Questions for Future Research

Introduction to Zeckendorf Decompositions

The Zeckendorf Decomposition

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- **Example**: 108 = 89 + 13 + 5 + 1
- **Example**: 2018 = 1597 + 377 + 34 + 8 + 2

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- Uniqueness holds because the largest number we can decompose using $\{F_1, \dots, F_{k-1}\}$ is less than F_k; hence any decomposition of $n \geq F_k$ must use F_k.

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Remark: This only works if we start the Fibonaccis

$$1, 2, 3, 5, 8, \ldots$$

(Starting $1, 1$, or $0, 1$, would lose unique decomposition!)

Framework: Graph-Restricted Decompositions

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For example, the Zeckendorf decomposition rule corresponds to the graph G where adjacent vertices are connected.

The G-decomposition

Question: Does there always exist a good choice of sequence in which to G-decompose numbers?

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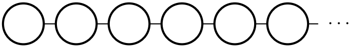
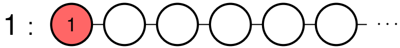
Example:
The G-sequence of this graph is the Fibonacci numbers

G : 

The G-sequence of the Zeckendorf graph



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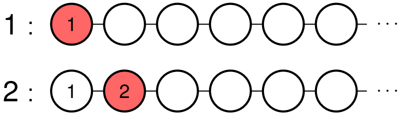

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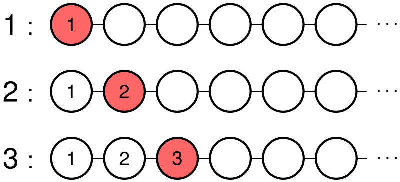

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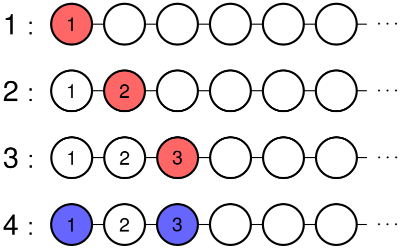

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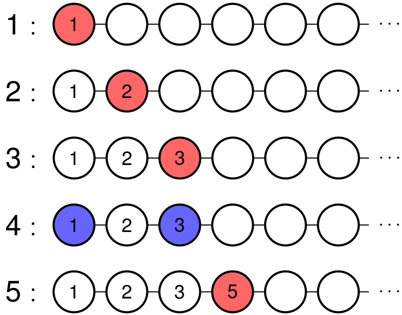

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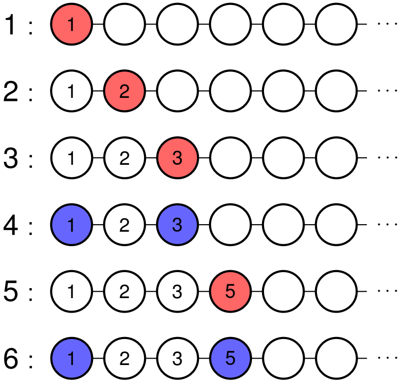

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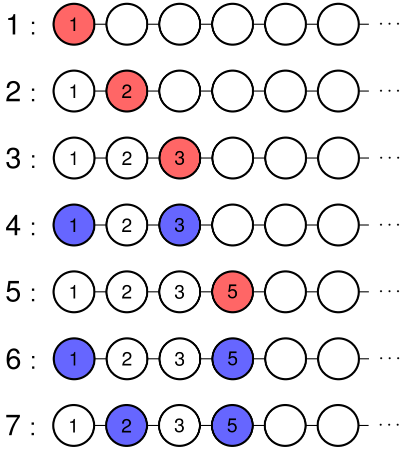

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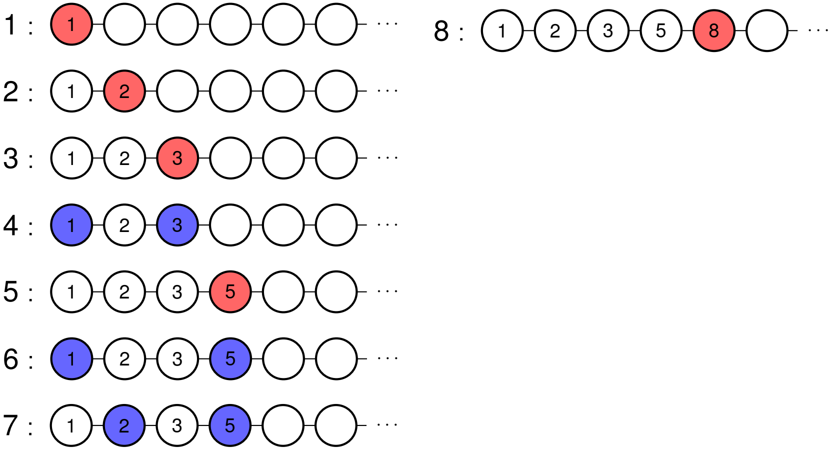

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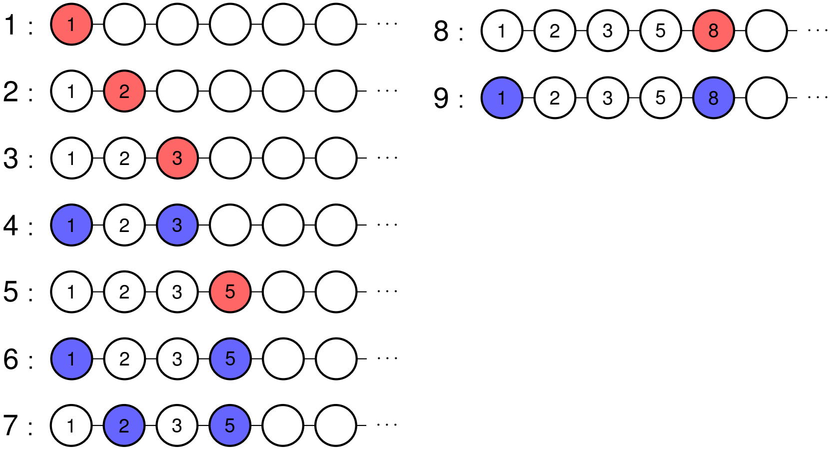

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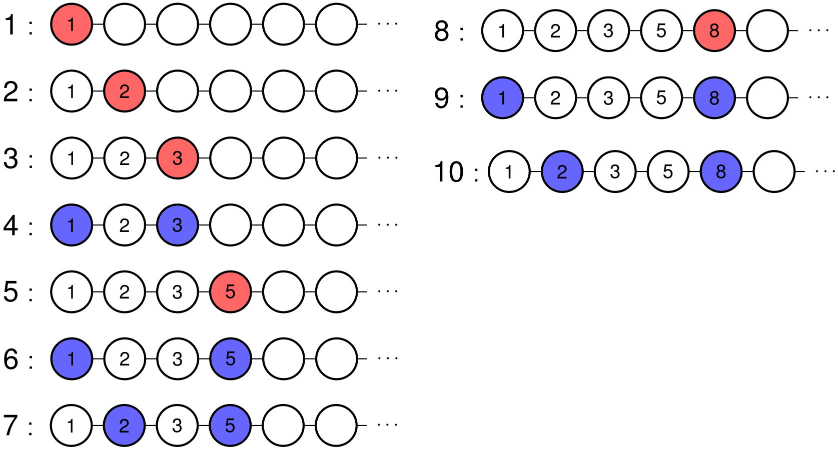

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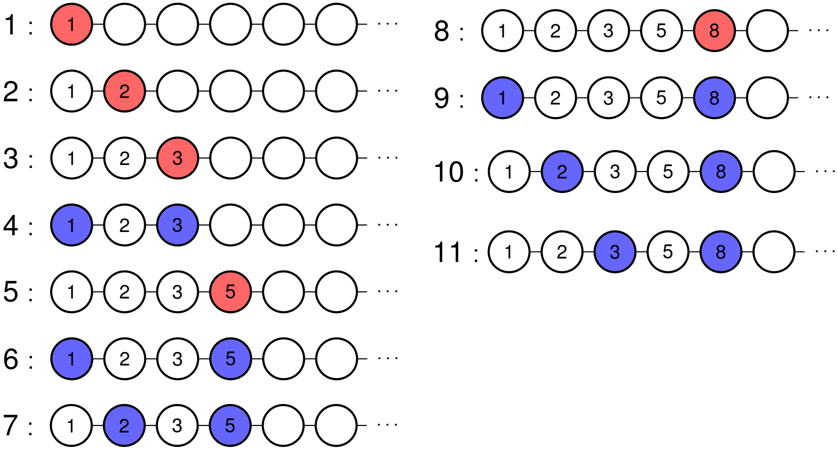

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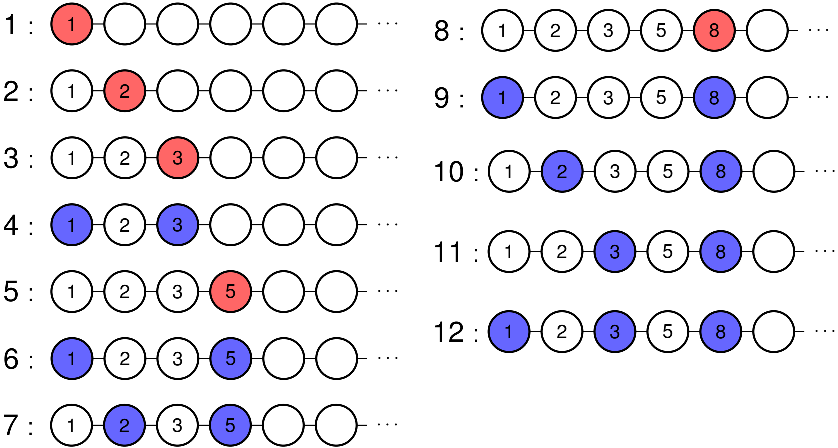

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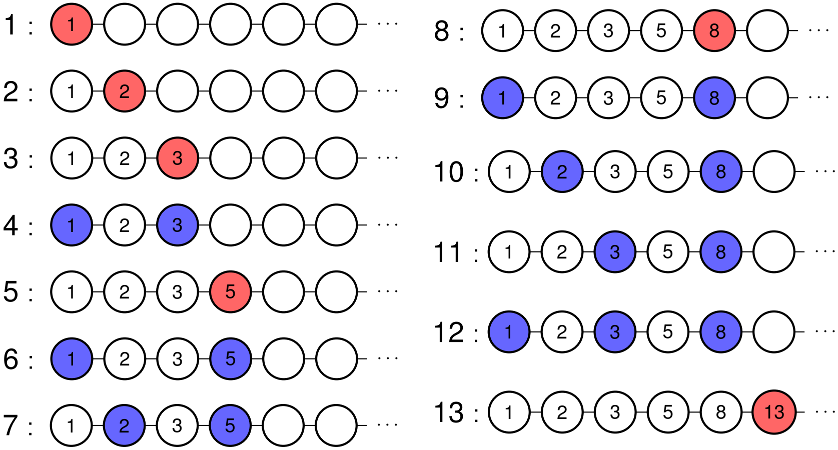

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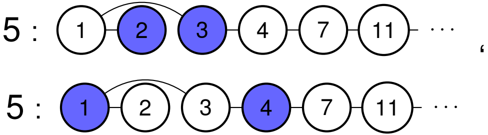

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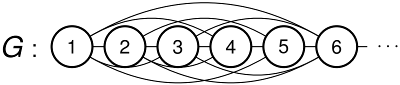
From now on, when we say G-decomposition, we mean G-decomposition in the G-sequence.

Examples

- Naturals



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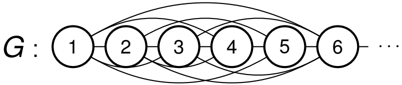

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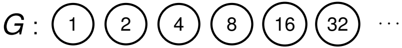

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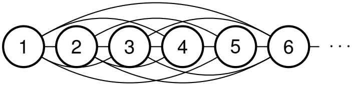

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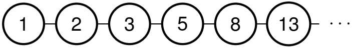

A Lot of Past Work is Special Cases!

- ***f*-decompositions** – more on these later
  (Demontigny et al., 2013)
- **Kentucky sequence**
  (Catral et al., 2014)
- **Quilt sequence**
  (Catral et al., 2016)
- **The Zeckendorf lattice**
  (Chen et al., 2018)

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Given a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ we say that if a_n is in an f**-decomposition**, the decomposition cannot contain the $f(n)$ terms immediately before a_n in the sequence.

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f-**decompositions come from graphs**:
an f-decomposition is the G-decomposition resulting from the graph G whose nth vertex is connected to the $f(n)$ vertices immediately before it.

New behavior

...Our framework will also help us see new behavior!

Results on Decomposition Behavior

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We produce a sufficient condition for uniqueness.

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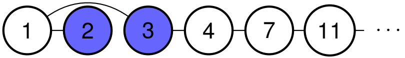
Non-example: $5 \in A_3^G$ but $5 > a_4^G = 4$



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This theorem shows that our uniformity condition is equivalent to the f-decompositions introduced by Demontigny, et al.

(Our framework has helped justify their definition, and gives a new perspective from which to ask questions.)

Uniformity implies Uniqueness

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Corollary

If G is uniform, then G-decompositions are unique.

Nice Properties of Uniform Graphs

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Corollary

If G, H are uniform graphs and H is a subgraph[a] of G then for all $k \in \mathbb{N}$, $a_k^H \geq a_k^G$.

[a] fewer edges, same vertices

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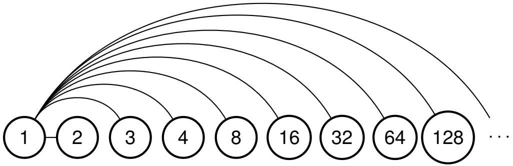
G : 

Distributions of Number of Summands

Counting Summands

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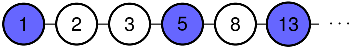

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The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

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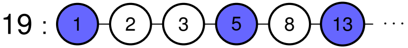

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We would also like to know what the distribution of the number of summands looks like.

Past Results

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Theorem (GCKMSSWY 2018)

As $n \to \infty$, the distribution of numbers of summands in a large class of mixed-radix decompositions is Gaussian.

New Results

Theorem

If G is a subgraph of the Zeckendorf graph, then it produces a Gaussian summand distribution.

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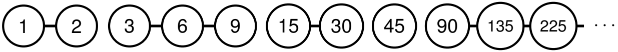
Example: a subgraph of the Zeckendorf Graph

G :



Gaussianity

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Open Question: which graphs G do we expect to produce Gaussian distributions of summands?

Non-Gaussianity

Theorem

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which doubles after it reaches $4^k/2$, and otherwise increases by 1.

$1, \mathbf{2}, 4, 5, 6, 7, \mathbf{8}, 16, 17, \dots, \mathbf{32}, 64, 65, \dots, \mathbf{128}, \dots$

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This produces a non-Gaussian distribution because it switches between *arithmetic* and *geometric* behavior, so a positive proportion of numbers require 1 summand, a smaller positive proportion require 2, and so on.

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It has a geometric distribution of number of summands:

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$$\left(\frac{2}{3}\right)^n \text{ of integers use } n \text{ summands.}$$

Connection to Growth Rate?

The previous example of non-Gaussian behavior have linear asymptotic growth rate, while most examples of Gaussianity exhibit exponential or near-exponential growth.

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Conjecture

If a uniform graph gives a sequence with exponential growth rate, then it produces a summand distribution which is Gaussian.

Questions for Future Research

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Sequences to Graphs

- How can you tell if a sequence is the G-sequence for some graph?
- Is there an algorithm which takes in a sequence and spits out a graph G which generates it, if one exists?
- For which sequences is there a unique graph which generates them?

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Growth Rates

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- What growth rates can G-sequences have?

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Growth Rates

- What growth rates can G-sequences have?
- Can we relate growth rate to edge density (or a different measure of how connected G is)?

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- Which graphs G give Gaussian distributions of summands?

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- Which graphs G give Gaussian distributions of summands?
- Is their a critical growth rate which ensures Gaussianity?

Questions for Future Research

Gaussianity

- Which graphs G give Gaussian distributions of summands?
- Is their a critical growth rate which ensures Gaussianity?
- Can we find a non-Gaussian distribution whose mean goes to infinity?

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Thank you!