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Thermal evolution of a pulsating neutron star

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ABSTRACT

We have derived a set of equations to describe the thermal evolution of a neutron star which undergoes small-amplitude radial pulsations. We have taken into account, within the framework of the general theory of relativity, the pulsation damping due to the bulk and shear viscosity and the accompanying heating of the star. The neutrino emission of a pulsating non-superfluid star and its heating due to the bulk viscosity are calculated assuming that both processes are determined by the non-equilibrium modified Urca process. Analytical and numerical solutions to the set of equations of the stellar evolution are obtained for linear and strongly non-linear deviations from beta-equilibrium. It is shown that a pulsating star may be heated to very high temperatures, while the pulsations damp very slowly with time as long as the damping is determined by the bulk viscosity (a power-law damping over 100–1000 yr). The contribution of the shear viscosity to the damping becomes important in a rather cool star with a low pulsation energy.

Key words: stars: evolution – stars: neutron – stars: oscillations.

1 INTRODUCTION

Dissipation processes play an important role in neutron star physics; for instance, they determine the damping of stellar pulsations (see, e.g., Cutler, Lindblom & Splinter 1990). Pulsations may be excited during star formation or during the evolution of the star under the action of external perturbations or internal instabilities. The instabilities arising in a rotating star from the emission of gravitational waves may be suppressed by dissipation processes. This affects the maximum rotation frequency of neutron stars and creates problems in the detection of gravitational waves (see, e.g., Zdunik 1996; Andersson & Kokkotas 2001; Lindblom 2001; Arras et al. 2003).

The joint thermal and pulsational evolution of neutron stars was studied long ago (e.g. Finzi & Wolf 1968 and references therein). Naturally, it was done with a simplified physics input and under restricted conditions (Section 4). However, later, while estimating the characteristic times of pulsational damping, one usually ignored the temporal evolution of the stellar temperature (see, e.g., Cutler & Lindblom 1987; Cutler et al. 1990), which led to an exponential damping. This is not always justified because the parameters defining the damping rate, e.g. the bulk and shear viscosity coefficients, are themselves temperature-dependent.

Clearly, the temperature variation can be neglected if the characteristic damping time $\tau \ll t_{\rm cool}$ and $E_{\rm puls} \ll E_{\rm th}$, where $t_{\rm cool}$ is the characteristic time of neutron star cooling, while $E_{\rm puls}$ and $E_{\rm th}$ are the pulsational and thermal energies, respectively. We will show that these conditions are violated over a wide range of initial temperatures and pulsation amplitudes.

This paper presents a self-consistent calculation of the dissipation of radial pulsations, taking into account the thermal evolution of a non-superfluid neutron star, the core of which consists of neutrons (n), protons (p) and electrons (e). It extends the consideration by Finzi & Wolf (1968) (see Section 4 for details). We consider two dissipation mechanisms: one is via the *non-linear* (in the pulsation amplitude) bulk viscosity in the stellar core and the other is due to the shear viscosity. We neglect other possible dissipation mechanisms, in particular, the damping of pulsations induced by the star magnetic field (as discussed in detail by McDermott et al. 1984 and by McDermott, van Horn & Hansen 1988). The magnetic field is assumed to be low.

2 EIGENFUNCTIONS AND EIGENFREQUENCIES OF NON-DISSIPATIVE RADIAL PULSATIONS

Here we discuss briefly radial pulsations of a neutron star, ignoring energy dissipation. This problem was first considered by Chandrasekhar (1964), and we will refer to his results. The metric for a spherically symmetric star, which experiences radial pulsations, can be written as

$$ds^2 = -e^{\nu}\, dt^2 + r^2\, d\Omega^2 + e^{\lambda}\, dr^2, \qquad (1)$$

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where r and t are the radial and time coordinates and $\mathrm{d}\Omega$ is a solid angle element in a spherical frame with the origin at the stellar centre. Here and below, we use the system of units, in which light velocity $c = 1$. The functions ν and λ depend only on r and t and can be written as $\nu(r,t) = \nu_0(r) + \delta\nu(r,t)$; $\lambda(r,t) = \lambda_0(r) + \delta\lambda(r,t)$. Here $\nu_0(r)$ and $\lambda_0(r)$ are the metric functions for an unperturbed (equilibrium) star, and $\delta\nu(r,t)$ and $\delta\lambda(r,t)$ are the metric perturbations due to the radial pulsations (described by equations 36 and 40 of Chandrasekhar 1964).

The radial pulsations can be found by solving the Sturm–Liouville problem (equation 59 of Chandrasekhar 1964). A solution gives eigenfrequencies of pulsations ω_k and eigenfunctions $\xi_k(r)$, where $\xi_k(r)$ is the Lagrangian displacement of a fluid element with a radial coordinate r. By neglecting the dissipation and the non-linear interaction between the modes (the pulsation amplitude is taken to be small, $|\xi_k(r)| \ll r$), we can write the general solution for a kth mode as $\xi(r, t) = \xi_k(r)\cos\omega_k t$. The boundary conditions for the Sturm–Liouville problem have the form $P(r = R + \xi(R, t)) = 0$, $\xi(0, t) = 0$, where $P(r, t)$ is the pressure and R is the unperturbed stellar radius.

We employ the equation of state of Negele & Vautherin (1973) in the stellar crust and the equation of state of Heiselberg & Hjorth-Jensen (1999) in the stellar core. The latter equation of state is a convenient analytical approximation of the equation of state of Akmal & Pandharipande (1997). For this equation of state, the most massive stable neutron star has a central density of $\rho_c = 2.76 \times 10^{15}$ g cm^{-3}, a circumferential radius of $R = 10.3$ km and a mass of $M = M_{\max} = 1.92\,\mathrm{M}_\odot$. The powerful direct Urca process of neutrino emission is open in the core of a star of mass $M > 1.83\,\mathrm{M}_\odot$.

An important parameter which enters the equation of radial pulsations is the adiabatic index γ. As the frequency of stellar pulsations is $\omega_k \gg 1/t_{\rm Urca}$, where $t_{\rm Urca}$ is the characteristic beta-equilibration time (see, e.g., Haensel, Levenfish & Yakovlev 2001; Yakovlev et al. 2001), the adiabatic index must be determined assuming the 'frozen' nuclear composition (see, e.g., Bardeen, Thorne & Meltzer 1966):

$$\gamma = \frac{\partial \ln P(n_{\rm b}, x_{\rm e})}{\partial \ln n_{\rm b}}, \tag{2}$$

where $n_{\rm b}$ is the baryon number density, $x_{\rm e} = n_{\rm e}/n_{\rm b}$, and $n_{\rm e}$ is the electron number density.

The relative radial displacement of matter elements in a pulsating star (in the absence of dissipation effects) will be described by a small parameter ε:

$$\varepsilon = \lim_{r\to 0} \xi_k(r)/r. \tag{3}$$

Thus, ε determines the normalization of the function $\xi_k(r)$.

Fig. 1 shows the dependence of $\xi_k(r)/r$ (artificially normalized such that $|\varepsilon| = 1$) on the distance to the stellar centre r for the first three modes with the frequencies $\omega_0 = 1.705 \times 10^4$ s^{-1} (solid line), $\omega_1 = 4.121 \times 10^4$ s^{-1} (long dashed line) and $\omega_2 = 5.950 \times 10^4$ s^{-1} (short dashed line), respectively. By way of illustration, we consider a model of a star of mass $M = 1.4\,\mathrm{M}_\odot$ ($R = 12.17$ km, $\rho_c = 9.26 \times 10^{14}$ g cm^{-3}). As expected, the fundamental mode is close to the homological solution $\xi_0(r) = r$. Introducing a normalization constant, we obtain $\xi_0(r) = \varepsilon\, r$. Therefore, for the fundamental mode, ε determines the amplitude of relative displacements of the pulsating stellar surface.

We will also need the pulsation energy, which can be calculated if we formulate, for example, the variational principle for the characteristic eigenvalue problem in question. For the kth radial mode,

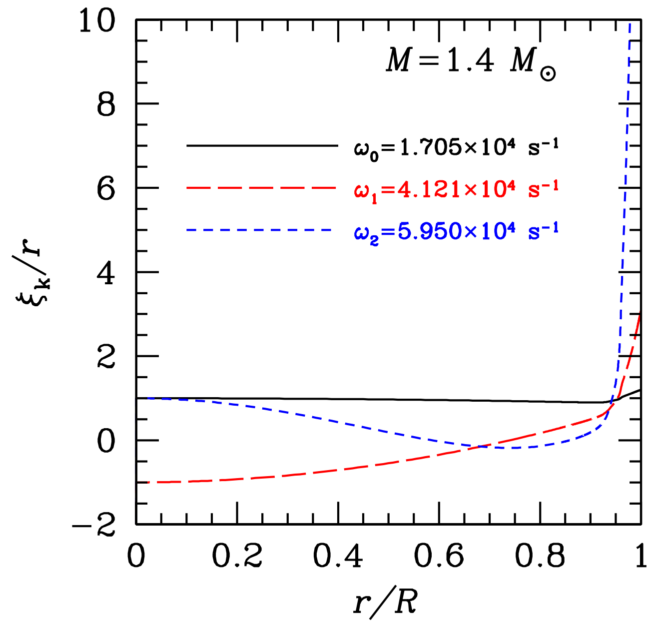

Figure 1. The parameter ξ_k/r normalized such that $|\varepsilon| = 1$, for the fundamental, first and second modes of radial stellar pulsations (solid, long-dashed and short-dashed lines, respectively) versus the dimensionless radial coordinate *r/R*.

we have (see, e.g., Meltzer & Thorne 1966)

$$E_{\rm puls} = \frac{1}{2}\int (P + \rho)\left[\mathrm{e}^{(\lambda_0-\nu_0)/2}\omega_k\,\xi_k\right]^2 \mathrm{e}^{\nu_0/2}\,\mathrm{d}V, \tag{4}$$

where ρ is the mass density and $\mathrm{d}V = 4\pi r^2 \mathrm{e}^{\lambda_0/2}\,\mathrm{d}r$ is the volume element measured in a comoving frame.

Taking account of the energy dissipation in the kth mode leads to a relatively slow damping of pulsations. In particular, we take for the Lagrangian displacement

$$\xi(r, t) = C_k(t)\,\xi_k(r)\cos\omega_k t, \tag{5}$$

where $C_k(t)$ is a slowly decreasing function of time (the characteristic dissipation time $\tau \gg 1/\omega_k$), which will be further termed the *pulsation amplitude*. The dissipation is assumed to be 'switched on' at the moment of time $t = 0$, at which the initial amplitude is

$$C_k(0) = 1. \tag{6}$$

From equation (4) the pulsation energy in the kth mode with dissipation is

$$E_{\rm puls}(t) = E_{\rm puls0}C_k^2(t). \tag{7}$$

Using equations (4) and (7), we can estimate the pulsation energy for the fundamental mode, $E_{\rm puls}(t) \sim 2 \times 10^{53}\,\omega_4^2\varepsilon^2 C_k^2(t)$ erg. The thermal energy of the star is $E_{\rm th} \sim (4\pi/3)R^3 c_T T \sim 10^{48} T_9^2$ erg, where $c_T \propto T$ is the specific (per unit volume) heat capacity of the stellar matter (see, e.g., Yakovlev, Levenfish & Shibanov 1999) and T_9 is the internal temperature of the star in units of 10^9 K, $\omega_4 = \omega_k/(10^4$ s$^{-1})$. These estimates show that there is a wide range of values of the parameters ε, $C_k(t)$ and T, at which $E_{\rm puls} \gtrsim E_{\rm th}$. In such a case, one should account, at least, for the stellar temperature evolution during the damping of pulsations.

3 NON-EQUILIBRIUM MODIFIED URCA PROCESS

The condition for beta-equilibrium in the stellar core has the form $\delta\mu(r,t) = \mu_{\rm n} - \mu_{\rm p} - \mu_{\rm e} = 0$, where μ_i is the chemical potential of particle species $i =$ n, p, e. Stellar pulsations lead to deviations from beta-equilibrium, $\delta\mu(r,t) \neq 0$, and hence, to the dissipation of the pulsation energy. The dissipation rate is determined by the processes tending to return the system to equilibrium. We suggest that the direct Urca process in the neutron star core is forbidden. Then the main process which determines the pulsation energy dissipation is the modified Urca process. In this section, we will discuss the non-equilibrium modified Urca process and obtain the relationship between the Lagrangian displacement $\xi(r,\,t)$ and the parameter $\delta\mu(r,t)$ that characterizes the local deviation from beta-equilibrium.

The non-equilibrium modified Urca process has been discussed since the end of the 1960s (see the classical paper by Finzi & Wolf 1968 and references therein). However, those old studies were accurate only qualitatively (see, e.g., Haensel 1992 and Section 4). Later the problem was reconsidered by several authors (see, e.g., Haensel 1992; Reisenegger 1995; Haensel et al. 2001). Below, references will primarily be made to the review of Yakovlev et al. (2001) since we employ a similar notation. The modified Urca process has the neutron and the proton branch, each including a direct and an inverse reaction:

$$\mathrm{n} + \mathrm{N} \rightarrow \mathrm{p} + \mathrm{N} + \mathrm{e} + \bar{\nu}_{\rm e}, \quad \mathrm{p} + \mathrm{N} + \mathrm{e} \rightarrow \mathrm{n} + \mathrm{N} + \nu_{\rm e}, \tag{8}$$

where N = n or p for the neutron or proton branch, respectively. In beta-equilibrium, the neutrino emissivities in these two channels are given by

$$Q_{\rm eq}^{(\rm n)} \approx 8.1 \times 10^{21} \left(\frac{m_{\rm n}^*}{m_{\rm n}}\right)^3 \left(\frac{m_{\rm p}^*}{m_{\rm p}}\right) \times \left(\frac{n_{\rm p}}{n_0}\right)^{1/3} T_9^8\, \alpha_{\rm n}\beta_{\rm n}\, \mathrm{erg\,cm^{-3}\,s^{-1}}, \tag{9}$$

$$Q_{\rm eq}^{(\rm p)} \approx Q_{\rm eq}^{(\rm n)} \left(\frac{m_{\rm p}^*}{m_{\rm n}^*}\right)^2 \frac{(p_{\rm Fe} + 3p_{\rm Fp} - p_{\rm Fn})^2}{8 p_{\rm Fe} p_{\rm Fp}}\, \Theta, \tag{10}$$

where $n_0 = 0.16\ \mathrm{fm}^{-3}$ is the nucleon number density in atomic nuclei; $n_{\rm p}$ is the proton number density; $m_{\rm n}$ and $m_{\rm p}$ are the masses of free neutrons and protons; $m_{\rm n}^*$ and $m_{\rm p}^*$ are the effective masses of neutrons and protons in dense matter; $p_{\rm Fe}$, $p_{\rm Fp}$ and $p_{\rm Fn}$ are, respectively, the Fermi momenta of electrons, protons and neutrons; and $\alpha_{\rm n}, \beta_{\rm n} \sim 1$ are the correction factors (for details, see Yakovlev et al. 2001). In equation (10) the function $\Theta = 1$ if the proton branch is allowed by momentum conservation ($p_{\rm Fn} < 3p_{\rm Fp} + p_{\rm Fe}$), and $\Theta = 0$ otherwise.

In beta-equilibrium, the direct and inverse reaction rates in equation (8) coincide, i.e. the matter composition does not change with time. The reactions involve only particles in the vicinity of their Fermi surfaces, with the energy $|\epsilon_i - \mu_i| \lesssim k_{\rm B}T$, where $i =$ n, p, e, and $k_{\rm B}$ is the Boltzmann constant. Therefore, the neutrino emissivity depends sensitively on the temperature, and the process cannot occur at $T = 0$. A drastically different situation arises in the presence of deviations from beta-equilibrium ($\delta\mu \neq 0$). The rates of the direct and inverse reactions become different, the system tends to equilibrium, and the matter composition changes; the process remains open even at $T = 0$.

Let Γ and $\bar{\Gamma}$ be the numbers of direct and inverse reactions of the modified Urca process per unit volume per unit time. The analytical expressions for $\Delta\Gamma = \bar{\Gamma} - \Gamma$ and for the neutrino emissivity $Q_{\rm non-eq} = Q^{(\rm n)} + Q^{(\rm p)}$ of the non-equilibrium modified Urca process were derived by Reisenegger (1995):

$$\Delta\Gamma = \frac{14\,680}{11\,513} \frac{Q_{\rm eq}}{k_{\rm B}T}\, y\, H(y), \tag{11}$$

$$Q_{\rm non\text{-}eq} = Q_{\rm eq}\, F(y), \tag{12}$$

where $Q_{\rm eq} = Q_{\rm eq}^{(\rm n)} + Q_{\rm eq}^{(\rm p)}$; and the functions $H(y)$ and $F(y)$ are given by

$$H(y) = 1 + \frac{189\pi^2 y^2}{367} + \frac{21\pi^4 y^4}{367} + \frac{3\pi^6 y^6}{1835}, \tag{13}$$

$$F(y) = 1 + \frac{22\,020\pi^2 y^2}{11\,513} + \frac{5670\pi^4 y^4}{11\,513} + \frac{420\pi^6 y^6}{11\,513} + \frac{9\pi^8 y^8}{11\,513}, \tag{14}$$

where $y \equiv \delta\mu/(\pi^2 k_{\rm B}T)$; the factor π^2 in the denominator is introduced to emphasize that the real variation scale of the functions $H(y)$ and $F(y)$ is $\delta\mu/(10k_{\rm B}T)$ (not just $\delta\mu/k_{\rm B}T$). It follows from equations (11) and (12) that there are two pulsation regimes. The regime with $\delta\mu \ll k_{\rm B}T$ ($y \ll 1$) will be referred to as *subthermal* and that with $\delta\mu \gg k_{\rm B}T$ ($y \gg 1$) as *suprathermal*. Relative displacements of fluid elements in both regimes are taken to be small ($\varepsilon \ll 1$). From these equations one can see that the values of $\delta\mu\,\Delta\Gamma$ and $Q_{\rm non-eq}$ in the suprathermal regime are independent of temperature.

Let us now find the relationship between the Lagrangian displacement $\xi(r,\,t)$ and the chemical potential difference $\delta\mu(r,\,t)$. The quantity $\delta\mu$ can be treated as a function of three thermodynamic variables, say, $n_{\rm b}$, $n_{\rm e}$ and T: $\delta\mu = \delta\mu(n_{\rm b}, n_{\rm e}, T)$. During pulsations, these variables will deviate from their equilibrium values $n_{\rm b0}$, $n_{\rm e0}$ and T_0 by $\Delta n_{\rm b}(r,t)$, $\Delta n_{\rm e}(r,t)$ and $\Delta T(r,t)$. Taking the deviations to be small, i.e. obeying the inequality $\varepsilon \ll 1$, one can write

$$\delta\mu(r,t) = \frac{\partial\delta\mu(n_{\rm b0}, n_{\rm e0}, T_0)}{\partial n_{\rm b0}} \Delta n_{\rm b}(r,t) + \frac{\partial\delta\mu(n_{\rm b0}, n_{\rm e0}, T_0)}{\partial n_{\rm e0}} \Delta n_{\rm e}(r,t) + \frac{\partial\delta\mu(n_{\rm b0}, n_{\rm e0}, T_0)}{\partial T_0} \Delta T(r,t). \tag{15}$$

The last term in equation (15) can be neglected because $\delta\mu(n_{\rm b0}, n_{\rm e0}, T_0)/T_0 \propto T_0$ and $\Delta T(r,\,t) \sim \Delta n_{\rm b}(r,\,t)\, T_0/n_{\rm b0}$ (see, e.g., Reisenegger 1995). Accordingly, for strongly degenerate matter ($\mu_i \gg k_{\rm B}T_0, i =$ n, p, e), this term is much smaller than the first two terms. The temperature T will further denote an 'average' temperature T_0 and its oscillations around the equilibrium value will be neglected.

The form of the functions $n_{\rm b}(r,\,t)$ and $n_{\rm e}(r,\,t)$ can be found from the continuity equations for baryons and electrons:

$$\left(n_{\rm b}u^\alpha\right)_{;\alpha} = 0, \tag{16}$$

$$\left(n_{\rm e}u^\alpha\right)_{;\alpha} = \Delta\Gamma, \tag{17}$$

where $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}s$ is the velocity four-vector of the pulsating matter. Note that the source $\Delta\Gamma$ in the continuity equation for electrons is responsible for beta-relaxation processes.

Writing explicitly the covariant derivatives in equations (16) and (17) in the metric (1) and neglecting all terms which are quadratic and higher order in $\xi(r,\,t)$, one obtains

$$\frac{\partial n_{\rm b}}{\partial t} + \frac{\mathrm{e}^{\nu_0/2}}{r^2} \frac{\partial}{\partial r}\left[n_{\rm b0} r^2 \mathrm{e}^{-\nu_0/2} \frac{\partial\xi(r,t)}{\partial t}\right] = 0, \tag{18}$$

$$\frac{\partial n_{\rm e}}{\partial t}+\frac{{\rm e}^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left[n_{\rm e0}r^2{\rm e}^{-\nu_0/2}\frac{\partial\xi(r,t)}{\partial t}\right]=\Delta\Gamma{\rm e}^{\nu_0/2}. \quad (19)$$

These expressions have been derived using equation (36) of Chandrasekhar (1964) for the correction $\delta\lambda(r, t)$ to the metric (see Section 2):

$$\delta\lambda(r,t)=-\xi(r,t)\,\frac{{\rm d}}{{\rm d}r}(\lambda_0+\nu_0). \quad (20)$$

Equation (18) is easily integrated and yields

$$\Delta n_{\rm b}(r,t)\equiv n_{\rm b}(r,t)-n_{\rm b0} = -\frac{{\rm e}^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left[n_{\rm b0}r^2{\rm e}^{-\nu_0/2}\,\xi(r,t)\right]. \quad (21)$$

The solution to equation (19) can be written as

$$\Delta n_{\rm e}(r,t)\equiv n_{\rm e}(r,t)-n_{\rm e0}=\Delta n_{\rm e0}(r,t)+\Delta n_{\rm e1}(r,t), \quad (22)$$

$$\Delta n_{\rm e0}(r,t)=-\frac{{\rm e}^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left[n_{\rm e0}r^2{\rm e}^{-\nu_0/2}\,\xi(r,t)\right], \quad (23)$$

where $\Delta n_{\rm e0}(r, t)$ describes variations of the electron number density ignoring beta-processes, while the function $\Delta n_{\rm e1}(r, t)$ describes variations determined by these processes. The latter function satisfies the equation

$$\frac{\partial\Delta n_{\rm e1}}{\partial t}=\Delta\Gamma\,{\rm e}^{\nu_0/2}. \quad (24)$$

Generally, the source $\Delta\Gamma$ is a complicated function of the electron number density $n_{\rm e}(r, t)$. We are, however, interested in the high-frequency limit, where $\omega_k \gg 1/t_{\rm Urca}$ (see Section 2). In that case, the source in the right-hand side of equation (19) is smaller than other terms. This means that changes in the electron number density due to beta-transformations are relatively small in a pulsating star (see, e.g., Haensel et al. 2001). Therefore, the small parameter $\Delta n_{\rm e1}$ can be omitted in equation (15).

By substituting the expressions for $\Delta n_{\rm b}$ and $\Delta n_{\rm e}$ from equations (21) and (23) into equation (15), we find the relationship between $\delta\mu(r, t)$ and $\xi(r, t)$:

$$\delta\mu(r,t)=-\frac{\partial\delta\mu(n_{\rm b0},x_{\rm e0})}{\partial n_{\rm b0}}n_{\rm b0}\,\frac{{\rm e}^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}[r^2{\rm e}^{-\nu_0/2}\,\xi(r,t)]. \quad (25)$$

Note that the partial derivative with respect to $n_{\rm b0}$ is taken at constant $x_{\rm e0} = n_{\rm e0}/n_{\rm b0}$. Using equation (25), we can express the parameter $y = \delta\mu/(\pi^2 k_{\rm B} T)$, as well as $\Delta\Gamma$ and $Q_{\rm non-eq}$ (see equations 11 and 12), through the Lagrangian displacement $\xi(r, t)$. The relationship between $\delta\mu(r, t)$ and $\xi(r, t)$ for non-radial pulsations can be derived in a similar way.

4 THE EQUATIONS OF STELLAR THERMAL EVOLUTION AND PULSATION DAMPING OUT OF BETA-EQUILIBRIUM

The thermal balance equation for a pulsating neutron star will be derived, taking into account three dissipation mechanisms: the shear viscosity in the core, the non-equilibrium beta-processes in the core and heat conduction. The equations of relativistic fluid dynamics to describe energy–momentum conservation are written as

$$T^{\alpha\beta}_{;\beta}=-Q_\nu\,u^\alpha, \quad (26)$$

where Q_ν is the total neutrino emissivity of all processes (including the non-equilibrium modified Urca process described by equation 12); $T^{\alpha\beta}$ is the energy–momentum tensor to be written as (see, e.g., Weinberg 1971):

$$T^{\alpha\beta}=Pg^{\alpha\beta}+(P+\rho)\,u^\alpha u^\beta + \Delta T^{\alpha\beta}_{\rm shear}+\Delta T^{\alpha\beta}_{\rm cond}, \quad (27)$$

$$\Delta T^{\alpha\beta}_{\rm shear}=-\eta\,H^{\alpha\gamma}H^{\beta\delta}\left(u_{\gamma\,;\delta}+u_{\delta\,;\gamma}-\frac{2}{3}\,g_{\gamma\delta}\,u^\lambda{}_{;\lambda}\right), \quad (28)$$

$$\Delta T^{\alpha\beta}_{\rm cond}=-\kappa(H^{\alpha\gamma}u^\beta+H^{\beta\gamma}u^\alpha)\left(T_{;\gamma}+T\,u_{\gamma\,;\delta}\,u^\delta\right), \quad (29)$$

where $g^{\alpha\beta}$ is the metric tensor, η is the shear viscosity coefficient, κ is the thermal conductivity and $H^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ is the projection matrix. In this paper we use $\eta = \eta_{\rm e}$, where the electron shear viscosity $\eta_{\rm e}$ in the stellar core is taken from Chugunov & Yakovlev (2005). We neglect the shear viscosity of neutrons (the proton shear viscosity is even smaller, see Flowers & Itoh 1979) because it depends strongly on the nuclear interaction model and the many-body theory employed. A similar problem for heat conduction was discussed by Baiko, Haensel & Yakovlev (2001). The neutron shear viscosity is comparable to the electron shear viscosity (Flowers & Itoh 1979), but it cannot change our results qualitatively.

The use of equations (16), (17) and (26) together with the second law of thermodynamics (${\rm d}\rho = \mu_{\rm n}\,{\rm d}n_{\rm n} + \mu_{\rm p}\,{\rm d}n_{\rm p} + \mu_{\rm e}\,{\rm d}n_{\rm e} + T\,{\rm d}S$) can yield the continuity equation for the entropy in the neutron star core (see, e.g., Landau & Lifshitz 1959; Weinberg 1971):

$$(Su^\alpha)_{;\alpha}=(Q_{\rm bulk}+Q_{\rm shear}+Q_{\rm cond}-Q_\nu)/k_{\rm B}T, \quad (30)$$

$$Q_{\rm bulk}=\delta\mu\,\Delta\Gamma,\quad Q_{\rm shear}=\left(\Delta T^{\alpha\beta}_{\rm shear}\right)_{;\beta}u_\alpha, \quad (31)$$

$$Q_{\rm cond}=\left(\Delta T^{\alpha\beta}_{\rm cond}\right)_{;\beta}\,u_\alpha, \quad (32)$$

where S is the entropy density and $Q_{\rm bulk}$ is the pulsation energy dissipating into heat per unit volume per unit time owing to the non-equilibrium modified Urca process. The latter term can be interpreted as viscous dissipation due to an *effective bulk viscosity*. We will show below that at $\delta\mu \ll k_{\rm B}T$ it coincides with the term commonly considered by other authors (see, e.g., Sawyer 1989 or Haensel et al. 2001). The term $Q_{\rm shear}$ describes the dissipation of pulsation energy into heat due to the shear viscosity. The term $Q_{\rm cond}$ is generally responsible for heat diffusion in the star bulk and for the dissipation of pulsation energy due to heat conduction. Finally, Q_ν is the neutrino emissivity. In this work, the quantity $Q_{\rm cond}$ was calculated using an unperturbed metric [the metric (1) with $\nu = \nu_0$ and $\lambda = \lambda_0$] neglecting temperature variations over a pulsation period. The result coincides with the similar expression well known in the cooling theory of non-pulsating neutron stars (see, e.g., Thorne 1977; van Riper 1991). These assumptions are quite reasonable in the case of strongly degenerate matter. The damping due to heat conduction has been analysed by Cutler & Lindblom (1987) for the more general case of non-radial pulsations. The conclusion drawn by these authors was that the contribution of heat conduction to the dissipation of pulsation energy can be neglected.

For the spherically symmetric metric (1), the left-hand side of equation (30) can be rewritten as

$$(Su^\alpha)_{;\alpha}=\left\{\frac{\partial(S\,{\rm e}^{\lambda/2})}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2S\,{\rm e}^{\lambda/2}\frac{\partial\xi(r,t)}{\partial t}\right]\right\}{\rm e}^{-(\lambda+\nu)/2}. \quad (33)$$

In our further treatment, we will use the *isothermal approximation*, in which the redshifted internal temperature is taken to be constant over the star bulk: $\widetilde{T} = T\,{\rm e}^{\nu/2} = $ constant. This approximation works well for a cooling star of age $t \gtrsim (10–50)$ yr (see, e.g., Yakovlev et al. 2001; Yakovlev & Pethick 2004). The isothermal approximation considerably simplifies calculations without any significant loss of accuracy (at least for the case of non-equilibrium modified Urca processes). Using equation (33) and the integral form of equation (30), and averaging over a pulsation period, we arrive at the thermal balance equation:

$$\frac{\mathrm{d}E_{\mathrm{th}}}{\mathrm{d}t} \equiv C_T \frac{\mathrm{d}\widetilde{T}}{\mathrm{d}t} = -L_{\mathrm{phot}} - L_\nu + W_{\mathrm{bulk}} + W_{\mathrm{shear}}, \qquad (34)$$

$$C_T = \int c_T \, \mathrm{d}V, \qquad (35)$$

$$L_{\mathrm{phot}} = 4\pi R^2 \sigma T_{\mathrm{s}}^4 \, \mathrm{e}^{\nu_0(R)}, \quad L_\nu = \int \overline{Q}_\nu \, \mathrm{e}^{\nu_0} \, \mathrm{d}V, \qquad (36)$$

$$W_{\mathrm{bulk}} = \int \overline{Q}_{\mathrm{bulk}} \, \mathrm{e}^{\nu_0} \, \mathrm{d}V, \quad W_{\mathrm{shear}} = \int \overline{Q}_{\mathrm{shear}} \, \mathrm{e}^{\nu_0} \, \mathrm{d}V, \qquad (37)$$

where L_{phot} and L_ν are the redshifted photon and neutrino luminosities of the star; W_{bulk} and W_{shear} denote the heat released in the star per unit time owing to the bulk and shear viscosities, respectively; σ is the Stefan–Boltzmann constant; T_{s} is the effective surface temperature (the relationship between the surface and internal temperatures is reviewed, for example, by Yakovlev & Pethick 2004). The upper horizontal line denotes averaging over a pulsation period. In deriving equation (34), we neglected the terms of the order of $\sim T^2 \varepsilon^2$, as compared with those of $\sim T^2$, in the left-hand side of the equality. The term Q_{cond} in equation (30) leads to the appearance of L_{phot} in equation (34).

The rate of the heat release due to the shear viscosity is

$$\overline{Q}_{\mathrm{shear}} = \frac{\eta}{3} \frac{\mathrm{e}^{-\nu_0}}{r^2} \overline{\left(-2r \frac{\partial^2 \xi(r,t)}{\partial r \partial t} + 2 \frac{\partial \xi(r,t)}{\partial t} + r \frac{\partial \xi(r,t)}{\partial t} \frac{\mathrm{d}\nu_0}{\mathrm{d}r} \right)^2}$$
$$= \frac{\eta}{6} \omega_k^2 C_k^2(t) \frac{\mathrm{e}^{-\nu_0}}{r^2} \left(-2r \frac{\mathrm{d}\xi_k}{\mathrm{d}r} + 2\xi_k + r\xi_k \frac{\mathrm{d}\nu_0}{\mathrm{d}r} \right)^2. \qquad (38)$$

Among the processes contributing to the neutrino emissivity Q_ν, the only process for which the emissivity $Q_{\mathrm{non\text{-}eq}}$ can vary dramatically over a pulsation period is the modified Urca process (we assume that the direct Urca process is forbidden). The expression for $\overline{Q}_{\mathrm{non\text{-}eq}}$ is obtained from equation (12) by averaging over the pulsation period $P = 2\pi/\omega_k$ allowing for $y(r, t) = y_0 \cos(\omega_k t)$, where y_0 is a slowly varying function of time:

$$\overline{Q}_{\mathrm{non\text{-}eq}} = Q_{\mathrm{eq}} \left(1 + \frac{11\,010\pi^2 y_0^2}{11\,513} + \frac{8505\pi^4 y_0^4}{46\,052} + \frac{525\pi^6 y_0^6}{46\,052} + \frac{315\pi^8 y_0^8}{1473\,664} \right), \qquad (39)$$

$$y_0 = \frac{C_k(t)}{\pi^2 k_{\mathrm{B}} T} \frac{\partial \delta\mu(n_{\mathrm{b}0}, x_{\mathrm{e}0})}{\partial n_{\mathrm{b}0}} n_{\mathrm{b}0} \frac{\mathrm{e}^{\nu_0/2}}{r^2} \frac{\partial}{\partial r} \left(r^2 \mathrm{e}^{-\nu_0/2} \xi_k \right). \qquad (40)$$

Similarly, equations (11) and (25) result in the expression for the heating rate produced by the dissipation of the pulsation energy due to deviations from beta-equilibrium:

$$\overline{Q}_{\mathrm{bulk}} = \frac{14\,680\pi^2}{11\,513} Q_{\mathrm{eq}} \times \left(\frac{y_0^2}{2} + \frac{567\pi^2 y_0^4}{2936} + \frac{105\pi^4 y_0^6}{5872} + \frac{21\pi^6 y_0^8}{46\,976} \right). \qquad (41)$$

Analogous expressions for the non-equilibrium direct Urca process are presented in the Appendix.

The quantities $\overline{Q}_{\mathrm{non\text{-}eq}}(y_0)$ and $\overline{Q}_{\mathrm{bulk}}(y_0)$ for the modified Urca process were calculated numerically by Finzi & Wolf (1968). Their results (their fig. 1) at $y_0 \lesssim 1$ are correct only qualitatively (Haensel 1992), although they become exact in the limit of $y_0 \gg 1$.

Using equations (39) and (41), one can easily find the neutrino emissivity and the viscous dissipation rate of the non-equilibrium modified Urca process for both subthermal or suprathermal pulsations (if they are small, i.e. $\varepsilon \ll 1$). The typical value $\overline{y}_0$ of the parameter y_0 in the stellar core can be estimated for the fundamental mode as

$$\overline{y}_0 \sim 100\,\varepsilon C_k(t)/T_9. \qquad (42)$$

Thus, we have derived the equation describing the thermal evolution of a pulsating neutron star. This equation depends on the current pulsation amplitude $C_k(t)$ and, hence, on the pulsation energy $E_{\mathrm{puls}}(t)$ (see equation 7). Let us now derive the equation to describe the evolution of the pulsation energy. In principle, it can be obtained from the 'pulsation equation' (59) of Chandrasekhar (1964) by taking into account the dissipation terms and considering them as small perturbations (generally non-linear in ξ_k). We have performed this derivation, but here we will present a much simpler derivation following from the energy conservation law. One should bear in mind that the pulsation energy dissipates due to the bulk and shear viscosities and is fully spent to heat the star. The corresponding terms have already been found for the thermal balance equation (34). The same terms, but with the opposite sign, should be valid for the damping equation which can thus be presented in the form:

$$\frac{\mathrm{d}E_{\mathrm{puls}}}{\mathrm{d}t} = -W_{\mathrm{bulk}} - W_{\mathrm{shear}}. \qquad (43)$$

The set of equations (34) and (43) has to be solved to obtain self-consistent solutions for the pulsation amplitude C_k and temperature $\widetilde{T}$ as a function of time.

Similar equations were formulated, analysed and solved numerically by Finzi & Wolf (1968) under some simplified assumptions. In particular, the authors neglected the pulsational damping due to the shear viscosity ($W_{\mathrm{shear}} = 0$). They used approximate expressions for L_ν and W_{bulk} (see above) and neglected the effects of general relativity. In addition, they used simplified models of neutron stars and stellar oscillations. However, their approach was quite sufficient to understand the main effects of the non-equilibrium modified Urca process on the thermal evolution of neutron stars and damping of their vibrations. We extend this consideration using the updated microphysics input with the proper treatment of the effects of the shear viscosity and general relativity.

We can generally write:

$$C_T = 10^{39}\, a_C\, \widetilde{T}_9 \,\mathrm{erg\,K^{-1}}, \quad L_{\nu 0} = 10^{40}\, a_L\, \widetilde{T}_9^8 \,\mathrm{erg\,s^{-1}},$$
$$E_{\mathrm{puls}} = 10^{53}\, a_P\, \omega_4^2 \varepsilon^2 C_k^2 \,\mathrm{erg}, \quad E_{\mathrm{th}} = C_T \widetilde{T}/2,$$
$$L_\nu = L_{\nu 0} \left(1 + a_1 \overline{y}_0^2 + a_2 \overline{y}_0^4 + a_3 \overline{y}_0^6 + a_4 \overline{y}_0^8 \right),$$
$$W_{\mathrm{shear}} = 10^{38}\, a_S\, \omega_4^2 \overline{y}_0^2 \,\mathrm{erg\,s^{-1}},$$
$$W_{\mathrm{bulk}} = L_{\nu 0} \left(\frac{2}{3} a_1 \overline{y}_0^2 + \frac{4}{3} a_2 \overline{y}_0^4 + 2 a_3 \overline{y}_0^6 + \frac{8}{3} a_4 \overline{y}_0^8 \right),$$
$$\overline{y}_0 \equiv 10^2\, \varepsilon C_k / \widetilde{T}_9, \qquad (44)$$

where $\widetilde{T}_9 = \widetilde{T}/(10^9\mathrm{K})$; $L_{\nu 0}$ is the neutrino luminosity of a non-pulsating star; $a_C, a_L, a_P, a_S, a_1, \ldots, a_4$ are dimensionless factors which depend on a stellar model and on a pulsation mode. For our model of a neutron star with $M = 1.4\,\mathrm{M}_\odot$ (the equation of state of

Heiselberg & Hjorth-Jensen 1999) and for the fundamental pulsation mode we have obtained $a_C = 1.88$, $a_L = 5.34$, $a_P = 1.81$, $a_S = 4.75$, $a_1 = 9.18$, $a_2 = 20.0$, $a_3 = 15.0$, $a_4 = 3.61$.

5 ANALYTICAL SOLUTIONS AND LIMITING CASES

Before presenting numerical solutions to equations (34) and (43), let us point out general properties of the solutions and consider the limiting cases. Numerical values will be given for the fundamental mode and for the above model of the star with $M = 1.4\ \mathrm{M}_\odot$.

Equation (43) describes the damping of pulsations due to the bulk and shear viscosities. In the present problem, there are no instabilities that could amplify stellar pulsations. In contrast, the thermal balance equation (34) permits both stellar cooling (at $L_\nu + L_{\rm phot} > W_{\rm bulk} + W_{\rm shear}$) and heating due to the viscous dissipation of the pulsation energy (at $L_\nu + L_{\rm phot} < W_{\rm bulk} + W_{\rm shear}$).

One may expect qualitatively different solutions in the subthermal ($\overline{y}_0 \ll 1$) and suprathermal ($\overline{y}_0 \gg 1$) regimes. For the former, we have $E_{\rm puls} \ll E_{\rm th}$, while for the latter $E_{\rm puls} \gg E_{\rm th}$.

5.1 The modified Urca regime

A sufficiently hot star has $L_{\rm phot} \ll L_\nu$ and $W_{\rm shear} \ll W_{\rm bulk}$. Then the evolution of pulsations and the thermal evolution of the star are totally determined by the non-equilibrium modified Urca process. The main features of this *modified Urca regime* were analysed by Finzi & Wolf (1968). We present this analysis using a more accurate approach (see above).

This regime is conveniently studied by analysing the evolution of $\widetilde{T}(t)$ and $\overline{y}_0(t)$. Neglecting $L_{\rm phot}$ and $W_{\rm shear}$, we can rewrite equations (34) and (43) as

$$\frac{2E_{\rm th}}{\overline{y}_0}\frac{{\rm d}\overline{y}_0}{{\rm d}t} = L_\nu - W_{\rm bulk}\left(1 + \frac{E_{\rm th}}{E_{\rm puls}}\right) = \widetilde{T}^8\, A(\overline{y}_0), \tag{45}$$

$$\frac{2E_{\rm th}}{\widetilde{T}}\frac{{\rm d}\widetilde{T}}{{\rm d}t} = -L_\nu + W_{\rm bulk} = \widetilde{T}^8\, B(\overline{y}_0), \tag{46}$$

where $E_{\rm th}/E_{\rm puls} = a_C/(20\,\overline{y}_0^2\, a_P\, \omega_4^2)$; the functions $A(\overline{y}_0)$ and $B(\overline{y}_0)$ are independent of $\widetilde{T}$. Their exact form is easily deduced from equation (44). One immediately has ${\rm d}\ln\overline{y}_0/{\rm d}\ln\widetilde{T} = A(\overline{y}_0)/B(\overline{y}_0)$, which allows one (in principle) to obtain the relation between $\overline{y}_0$ and $\widetilde{T}$ in an integral form. Equations (45) and (46) have two special solutions.

The first solution is obvious and refers to an ordinary non-vibrating [$\overline{y}_0(t) \equiv 0$] cooling neutron star. In this case

$$\widetilde{T}(t) = \widetilde{T}(0)\Big/\left[1 + 6\beta_0 \widetilde{T}_9^6(0)\, t\right]^{1/6}, \tag{47}$$

where $\beta_0 = a_L/(10^8\, a_C)$. We have $\beta_0 \approx 1/(1.12\ \text{yr})$, for our neutron star model.

The second solution is realized (Finzi & Wolf 1968) if the initial value $\overline{y}_0(0)$ satisfies the equation

$$\frac{L_\nu}{W_{\rm bulk}} = 1 + \frac{E_{\rm th}}{E_{\rm puls}} = 1 + \frac{a_C}{20\,\overline{y}_0^2\, a_P\, \omega_4^2}, \tag{48}$$

at which $A(\overline{y}_0) = 0$ and ${\rm d}\overline{y}(t)/{\rm d}t = 0$. In this case $\overline{y}_0(t)$ *remains constant* during the modified Urca stage. We will denote this specific value of $\overline{y}_0$ by $\overline{y}_{0L}$; it is equal to $\overline{y}_{0L} \approx 0.607$ for our neutron star model. In this limiting case $\widetilde{T}(t)$ and $C_k(t)$ are easily obtained from equation (34):

$$\widetilde{T}(t) = \widetilde{T}(0)\Big/\left[1 + 6\beta \widetilde{T}_9^6(0)\, t\right]^{1/6}, \tag{49}$$

$$C_k(t) = \widetilde{T}_9(t)\, \overline{y}_{0L}/10^2\varepsilon, \tag{50}$$

where $\beta = a_L(L_\nu - W_{\rm bulk})/(10^8\, a_C\, L_{\nu 0})$ [$\approx 1/(3.05\ \text{yr})$, for our model]. Thus, the internal stellar temperature $\widetilde{T}(t)$ and the pulsation amplitude $C_k(t)$ simultaneously decrease with time, leaving the suprathermality level constant, intermediate between the subthermal and suprathermal pulsation regimes. The decrease follows a power law (i.e. is non-exponential).

The thermal evolution of neutron stars in the two limiting cases is remarkably similar. If the star was born sufficiently hot [$\widetilde{T}(0) \gtrsim 10^9$ K] then within a few years after the birth the initial temperature becomes forgotten. For the non-vibrating star from equation (47) we have $\widetilde{T}_9^{(1)}(t) \approx (6\beta_0\, t)^{-1/6}$, while for the vibrating star from equation (49) we have $\widetilde{T}_9^{(2)}(t) \approx (6\beta\, t)^{-1/6}$. Thus, the vibrating star stays somewhat hotter, $\widetilde{T}^{(2)}(t)/\widetilde{T}^{(1)}(t) = (\beta_0/\beta)^{1/6}$ (≈ 1.18 for our model).

Once the two limiting solutions are obtained, all other solutions for the modified Urca regime become clear. If $\overline{y}_0(0) > \overline{y}_{0L}$, then $A(\overline{y}_0(0)) < 0$ and $\overline{y}_0(t)$ will tend to $\overline{y}_{0L}$ from above. If $\overline{y}_0(0) < \overline{y}_{0L}$, then $A(\overline{y}_0(0)) > 0$ and $\overline{y}_0(t)$ will tend to $\overline{y}_{0L}$ from below. After $\overline{y}_0(t)$ comes sufficiently close to $\overline{y}_{0L}$, the stellar evolution is approximately described by the limiting solution given by equations (49) and (50). Therefore, this limiting solution describes the *universal asymptotic behaviour of all vibrating neutron stars*.

5.2 The damping of oscillations by the shear viscosity

One may expect qualitatively different solutions for the damping due to the bulk viscosity ($W_{\rm bulk} \gg W_{\rm shear}$, a hot star) and the shear viscosity ($W_{\rm shear} \gg W_{\rm bulk}$, a cold star). Using equations (44), it is possible to show that the temperature $T_{\rm visc}$ separating these two regimes (and obeying the condition $W_{\rm bulk} \sim W_{\rm shear}$) is approximately equal to $T_{\rm visc} \sim 7 \times 10^8/(1 + \overline{y}_0^2)^{3/8}$ K. For the regime of damping due to shear viscosity ($T \ll T_{\rm visc}$), equation (43) reduces to a linear equation for $C_k(t)$, irrespective of the value of $\overline{y}_0$:

$$\frac{{\rm d}C_k(t)}{{\rm d}t} = -\frac{\alpha_{\rm shear}}{2\widetilde{T}_9^2} C_k(t), \tag{51}$$

where $\alpha_{\rm shear} \approx 3 \times 10^{-11}\ \text{s}^{-1} \sim 1/(1000\ \text{yr})$ is a constant factor. The solution to this equation shows an exponential damping, which is independent of $\overline{y}_0$:

$$C_k(t) = C_k(t_0)\exp\left[-\frac{\alpha_{\rm shear}}{2}\int_{t_0}^{t}\frac{{\rm d}t'}{\widetilde{T}_9^2(t')}\right]. \tag{52}$$

5.3 Subthermal pulsations

In this case ($y_0 \ll 1$), equation (41) can be reduced to

$$\overline{Q}_{\rm bulk} = \frac{14\,680\pi^2}{11\,513}\, Q_{\rm eq}\frac{y_0^2}{2} = \zeta\,\overline{\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2{\rm e}^{-\nu_0/2}\frac{\partial\xi(r,t)}{\partial t}\right)\right]^2} = \zeta\,\overline{\left(u^{\alpha}_{;\alpha}\right)^2}, \tag{53}$$

$$\zeta = \frac{14\,680}{11\,513\pi^2}\frac{Q_{\rm eq}}{(k_{\rm B}T)^2}\frac{n_{\rm b0}^2}{(\omega_k{\rm e}^{-\nu_0/2})^2}\left[\frac{\partial\delta\mu(n_{\rm b0}, x_{\rm e0})}{\partial n_{\rm b0}}\right]^2. \tag{54}$$

The quantity ζ can be treated as the bulk viscosity. Equation (54) coincides with the corresponding expression of Sawyer (1989) and Haensel et al. (2001). If the temperature remains constant during the damping, equations (7) and (43) yield an exponential fall of the pulsation amplitude $C_k(t)$, which is often discussed in the literature (see, e.g., Cutler et al. 1990).

According to equation (44), the subthermal regime is characterized by $L_\nu \approx L_{\nu 0} \gg W_{\rm bulk}$. In this case pulsations do not affect the neutrino luminosity, and the energy dissipation due to the bulk viscosity cannot produce a considerable stellar heating. The dissipation due to the shear viscosity is also too weak, $W_{\rm shear} \ll L_{\nu 0}$. For these reasons, the pulsations do not change significantly the thermal balance equation (34) and the thermal evolution of the star. At the neutrino cooling stage (when $L_{\nu 0} \gg L_{\rm phot}$, which happens at $t \lesssim 10^5$ yr) we obtain the well-known formula (47) for non-superfluid neutron stars that cool via the modified Urca process. It can be rewritten as (see, e.g., Yakovlev & Pethick 2004)

$$t = C_T \widetilde{T}/(6L_{\nu 0}) \sim 1\ {\rm yr}/\widetilde{T}_9^6. \tag{55}$$

This value of t can be considered as a characteristic cooling time $t_{\rm cool}$ of the star with internal temperature $\widetilde{T}$. For a hot star with $W_{\rm bulk} \gg W_{\rm shear}$ ($T \gg T_{\rm visc}$; the modified Urca regime), equation (43) gives the characteristic pulsation damping time

$$t_{\rm puls} \sim E_{\rm puls}/W_{\rm bulk} \sim t_{\rm cool}. \tag{56}$$

Therefore, the internal temperature $\widetilde{T}$ and the typical imbalance of the chemical potentials $\overline{\delta\mu}$ decrease with approximately the same characteristic time $t_{\rm cool}$ (see, e.g., Yakovlev et al. 2001). The parameter $\overline{y}_0 \propto \overline{\delta\mu}/\widetilde{T}$, which describes the 'level' of pulsations relative to the thermal 'level', should tend to the limiting value $\overline{y}_{0L}$ (Section 5.1). The damping of pulsations obeys the power law (rather than an exponential, as would be the case in the absence of thermal evolution). This is because the viscous damping rate depends strongly on temperature, $W_{\rm bulk} \propto \widetilde{T}^6$.

In a cooler star ($\widetilde{T} \ll T_{\rm visc}$, $W_{\rm shear} \gg W_{\rm bulk}$), the damping of subthermal pulsations is due to the shear viscosity and occurs, according to equation (52), more abruptly (exponentially), decreasing the pulsation level $\overline{y}_0$.

5.4 Suprathermal pulsations

In this case, the quantity $\overline{Q}_{\rm bulk}$ cannot be generally described by an expression of the type of equation (53). Strictly speaking, we cannot introduce a bulk viscosity ζ, but equation (41) adequately describes the rate of the pulsation energy dissipation due to the modified Urca process. Nevertheless, at least for radial suprathermal pulsations, the quantity $\overline{Q}_{\rm bulk}$ can be formally calculated from equation (53), as before, with the effective bulk viscosity ζ being given by equation (54) with an additional factor of $\overline{Q}_{\rm bulk}/\overline{Q}_{\rm bulk}(\overline{y}_0 \to 0)$. In the suprathermal regime, the effective bulk viscosity and the viscous dissipation rate appear to be much larger than in the subthermal regime, as was pointed out by Haensel, Levenfish & Yakovlev (2002). However, there is an omission in their equations (13)–(15) for the effective bulk viscosity in the suprathermal regime: the authors should have introduced an additional factor of $\sim(1+\overline{y}_0^2)$. This does not change their principal results qualitatively. Nevertheless, we stress that while analysing the damping of pulsations, one should account for the thermal evolution of the star. Accordingly, in the suprathermal regime equation (16) of Haensel et al. (2002) gives the characteristic time of the *non-exponential* damping.

The pulsation equation (43) in the suprathermal regime ($E_{\rm puls} \gg E_{\rm th}$, $\overline{y}_0 \gg 1$) at $W_{\rm bulk} \gg W_{\rm shear}$ ($\widetilde{T} \gg T_{\rm visc}$; the modified Urca regime) can be rewritten as

$$\frac{{\rm d}C_k^2(t)}{{\rm d}t} = -\alpha_{\rm bulk} C_k^8(t), \tag{57}$$

where $\alpha_{\rm bulk} \approx 3 \times 10^4\ \varepsilon^6/\omega_4^2\ {\rm s}^{-1}$ is a constant factor. Assuming $C_k(0) = 1$ we obtain

$$C_k(t) \approx (1 + 3\alpha_{\rm bulk} t)^{-1/6}. \tag{58}$$

This solution describes a slow (power-law) fall of the pulsation amplitude $\sim t^{-1/6}$ with the characteristic time $1/(3\,\alpha_{\rm bulk})$. In this regime, the stellar heating always dominates over the cooling, with the heating rate $W_{\rm bulk} \approx \frac{8}{3} L_\nu \propto \overline{\delta\mu}^8$ being nearly independent of temperature (determined by the imbalance of the chemical potentials $\overline{\delta\mu}$). This result was obtained by Finzi & Wolf (1968). The power-law decrease of $C_k(t)$ is associated with a strong dependence of $W_{\rm bulk}$ on $\overline{\delta\mu}$ (which mimics the dependence on T in the subthermal regime). The relative pulsation amplitude should decrease, and the star should evolve to the subthermal regime ($\overline{y}_0 \to \overline{y}_{0L}$; Section 5.1).

In a rather cool star, the damping of pulsations due to the shear viscosity dominates over the damping caused by the bulk viscosity ($W_{\rm shear} \gg W_{\rm bulk}$, $\widetilde{T} \lesssim T_{\rm visc}$). The shear viscous damping is exponential, according to equation (51), so that the star rapidly evolves to the subthermal regime.

6 RESULTS

Generally, the set of equations (34) and (43) has no analytical solution, and we have to solve it numerically. We have modified the isothermal version of our cooling code (for details see the review by Yakovlev et al. 1999) by including a block for solving the damping equation (43). Our code calculates the stellar surface temperature $T_{\rm s}^\infty$ (redshifted for a distant observer), as a function of time t, as well as $C_k(t)$. All of the computations presented in this section are for the fundamental mode of radial pulsations. Computations for higher modes will not lead to qualitatively different conclusions.

The left-hand panel of Fig. 2 shows the thermal evolution paths of a neutron star ($M = 1.4\ {\rm M}_\odot$), which differ in the initial internal temperature $\widetilde{T}(0) = \widetilde{T}_0$ and the initial relative amplitude of pulsations ε (see equation 3). The right-hand panel presents $C_k(t)$ curves for the same models. The dotted curve on the left-hand panel shows the cooling of a non-pulsating star (in the isothermal approximation). The circle indicates the observations of the Vela pulsar. References to the original observations can be found in Gusakov et al. (2004).

The solid lines in both panels of Fig. 2 are for the model with $\widetilde{T}_0 = 10^{10}$ K and $\varepsilon = 0.01$. This model describes a star which was born hot and strongly pulsating. The initial pulsation energy is about half that of its initial thermal energy, and the star is in an intermediate pulsation regime, between the supra- and subthermal regimes, with $\overline{\delta\mu} \sim k_{\rm B}T$. The heating due to viscous dissipation is not as fast as the neutrino cooling due to the non-equilibrium modified Urca process and the star is cooling down. The main contribution to the dissipation at the initial stage is produced by the bulk viscosity. The maximum difference between the surface temperatures of such a star and a non-pulsating star occurs at $t \lesssim 1000$ yr. During this period of time, $\overline{\delta\mu}$ remains of the order of $k_{\rm B}T$ ($\overline{y}_0 \approx \overline{y}_{0L}$). At $t \gtrsim 1000$ yr, the damping begins to be determined by the shear viscosity, which is not as temperature-dependent as the bulk viscosity. This leads to exponential damping in the subthermal regime ($E_{\rm puls}/E_{\rm th} \ll 1$; see the right-hand panel of Fig. 2).

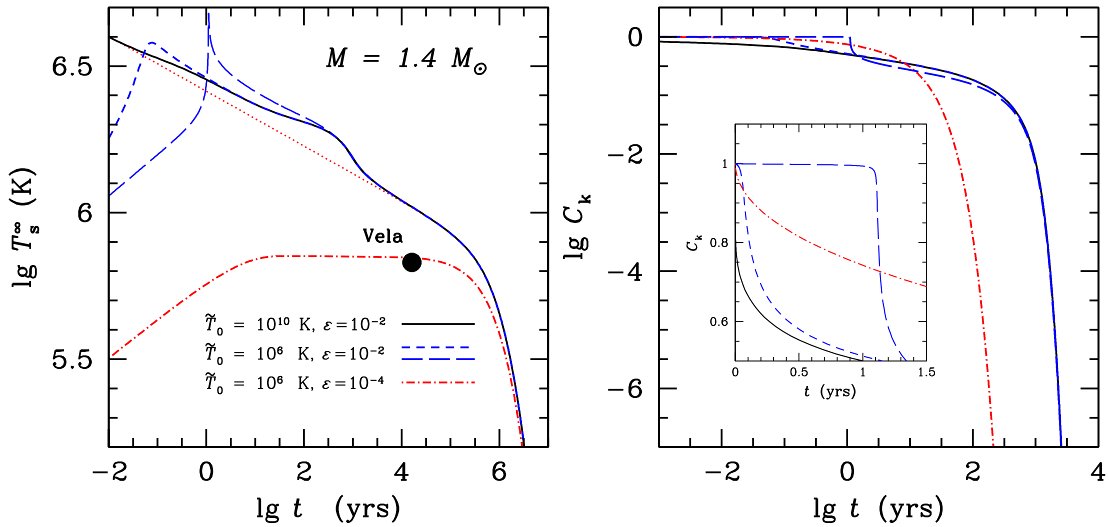

Figure 2. Temporal evolution of the surface temperature (left) and pulsation amplitude (right) for three initial conditions: $\widetilde{T}_0 = 10^{10}$ K, $\varepsilon = 0.01$ (solid lines); $\widetilde{T}_0 = 10^6$ K, $\varepsilon = 0.01$ (dashed lines); $\widetilde{T}_0 = 10^6$ K, $\varepsilon = 0.0001$ (dash-dotted lines). The short dashed lines denote a self-consistent calculation, the long dashed lines are calculated neglecting non-linear effects due to deviations from the beta-equilibrium (see the text). The dotted line on the left-hand panel shows the cooling of a non-pulsating star. The full circle denotes the observations of the Vela pulsar. The inset displays the pulsation damping at $t \lesssim 30$ yr in more detail.

The short dashed lines in Fig. 2 correspond to an initially cold star with $\widetilde{T}_0 = 10^6$ K and $\varepsilon = 0.01$. The initial ratio of the pulsation energy to the thermal energy is $E_{\rm puls0}/E_{\rm th0} \sim 5 \times 10^7$, i.e, the star pulsates in a strongly suprathermal regime. As follows from equation (44), at low temperatures we have $W_{\rm bulk} \ll W_{\rm shear}$, and the star is initially heated up by the shear viscosity. The heating due to the bulk viscosity starts to dominate only at $\widetilde{T} \gtrsim 5 \times 10^8$ K. After heating to $\widetilde{T} \approx 1.7 \times 10^9$ K in $t \sim 1$ month, the star appears in the intermediate regime with $\overline{\delta\mu} \sim k_{\rm B}T$ and begins to cool down. At $t \gtrsim 10$ yr, the star starts evolving along the same 'universal' path ($\overline{y}_0 \approx \overline{y}_{0L}$) as in the first model.

The long dashed curves are obtained for the same initial conditions but in the 'naive' approximation neglecting non-linear effects in non-equilibrium beta-processes. In particular, the neutrino luminosity is taken to be $L_\nu = L_{\nu 0}$ and the damping due to the bulk viscosity is determined by equations (53) and (54). One can see that this approximation leads to qualitatively incorrect results. The viscous heating during the first year after the pulsation excitation is much slower than in the scenario with non-linear effects, and the star heats up slowly. The neutrino luminosity is also lower, which enables the star to heat to higher temperatures. In fact, the slow damping due to the bulk viscosity does not decrease the pulsation amplitude during the first year.

The dash-and-dot lines in Fig. 2 refer to the cold star with $\widetilde{T}_0 = 10^6$ K and $\varepsilon = 0.0001$. The initial ratio of the pulsation and thermal energies is $E_{\rm puls0}/E_{\rm th0} \sim 5 \times 10^3$, which means that the star is initially in the suprathermal regime. Nevertheless, the pulsation energy $E_{\rm puls0} \sim 5 \times 10^{45}$ erg is insufficient to heat the star to a temperature at which the damping is determined by the bulk viscosity. For this reason, the pulsation energy is damped by the shear viscosity. The damping of pulsations takes ~100 yr (see the right-hand panel of Fig. 2). At $t \gtrsim 100$ yr the star cools via photon emission from the surface. It is clear from the left-hand panel that this model can, in principle, explain the surface temperature of a neutron star with the same thermal X-ray luminosity as the Vela pulsar but with a different history. For example, it may be an old and cold isolated neutron star, in which radial pulsations have been excited. Approximately 10 yr after the excitation, the star will be heated to the temperature of the Vela pulsar. In 100 yr, the pulsations will die out but the star will stay warm for $\sim 10^5$ yr before it starts cooling down noticeably. It should be emphasized that these results will not change if we take a lower initial temperature, e.g. $\widetilde{T}_0 = 10^4$ K. This star will also acquire a surface temperature of $T_{\rm s}^\infty \sim 7 \times 10^5$ K in a year and will emit in soft X-rays.

For a correct calculation of the pulsation damping, one should take into account the thermal evolution of the star. The evolutionary effects are especially important when the damping is determined by non-equilibrium beta-processes. They are relatively weak only in the subthermal regime, provided the damping is produced by shear viscosity.

These statements are also illustrated in Fig. 3 which shows the pulsation damping for a star with the initial internal temperature $\widetilde{T}(0) = 10^9$ K and the initial relative pulsation amplitude $\varepsilon = 0.01$. The initial ratio of the pulsation-to-thermal energy is $E_{\rm puls0}/E_{\rm th0} \sim 50$, indicating that the star is pulsating in a slightly suprathermal regime. The solid line is the result of a self-consistent solution of the thermal evolution and damping equations. The damping follows a power law for about 100 yr; afterwards the damping is determined by the shear viscosity and becomes exponential. The pulsations die out completely in ~1000 yr.

The dotted line in Fig. 3 shows the solution to the damping equation neglecting the thermal evolution, at a constant internal temperature $\widetilde{T} = \widetilde{T}(0)$. In this case, the pulsations are first damped by the bulk viscosity and then by the shear viscosity in ~30 yr.

The short dashed curve is obtained by taking into account the thermal evolution and damping, but neglecting the non-linear effects in non-equilibrium beta-processes. For about 100 yr, the damping is governed by the bulk viscosity; it follows a power law, but is slower than with the non-linear effects. Later, the shear viscosity becomes important, leading to exponential damping, nearly the same as with the non-linear effects.

Finally, the long dashed curve is calculated neglecting both the thermal evolution and the non-linear effects. As in the case with these effects (the dotted curve), the damping is steep (exponential),

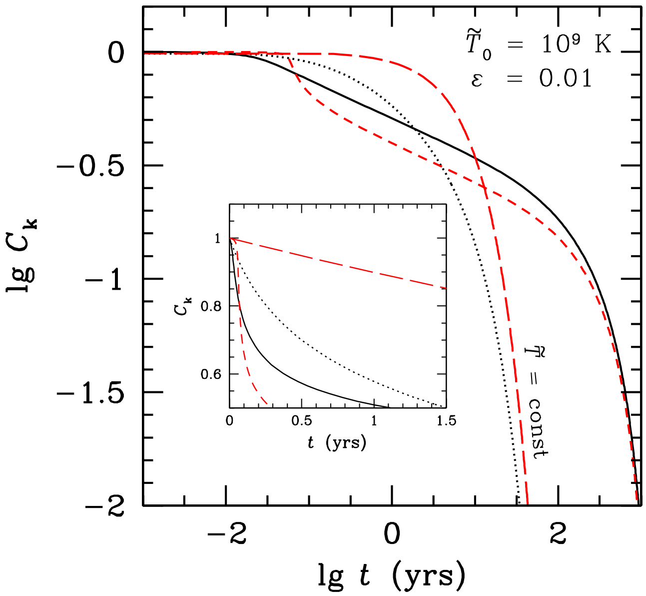

Figure 3. Temporal evolution of the pulsation amplitude for the initial conditions: $\widetilde{T}_0 = 10^9$ K, $\varepsilon = 0.01$. The solid line presents a consistent calculation taking account of the thermal evolution and non-linear deviations from beta-equilibrium (see the text); the dotted line shows the same but with the internal stellar temperature fixed; the short dashed line is the same as the solid line but without non-linear effects; the long dashed line is obtained by neglecting non-linear effects at the fixed temperature. The inset displays the pulsation damping at $t \lesssim 30$ yr in more detail.

taking about 30 yr, but occurs slightly more slowly (the long dashed curve is above the dotted curve).

7 SUMMARY

Extending the consideration of Finzi & Wolf (1968) we have analysed the thermal evolution of a non-superfluid star which undergoes small-amplitude radial pulsations. We have derived a set of equations to describe the thermal evolution and the damping of pulsations within the framework of general relativity. We have included the effects of non-linear deviations from beta-equilibrium in the modified Urca process on the neutrino luminosity and on the pulsation energy dissipation due to the bulk viscosity in the stellar core. We have also taken into account the dissipation due to the shear viscosity and the associated heating. A set of equations for the evolution of a neutron star with a nucleon core, in which the direct Urca process is forbidden, has been analysed and solved analytically and numerically.

We have shown that the evolution of a pulsating star depends strongly on the degree of non-linearity of the non-equilibrium modified Urca process and on the nature of the pulsation damping (the shear or bulk viscosity). In the non-linear regime, the star may be considerably heated by the pulsation energy dissipation but it is always cooled down in the linear regime. Pulsations of a hot star are damped by the bulk viscosity in both the linear and non-linear regimes, and this process is rather slow (a power law). In a cooler star, the damping is produced by the shear viscosity and goes much faster (exponentially). The characteristic times of damping of the fundamental mode lie within the range 100–1000 yr.

We have not discussed here the specific damping mechanism via the ambipolar diffusion of electrons and protons relative to neutrons when the averaged (over the period) chemical composition of the stellar matter remains constant in time. As far as we know, this mechanism of pulsation damping has not been analysed in the literature. However, it may be as efficient as the damping by the shear viscosity, at least in the suprathermal regime. We will consider this problem in a separate publication.

The analysis presented here is based on a simplified model. In particular, if the direct Urca process is open in the stellar core or if the core contains hyperons or quarks, the bulk viscosity can be many orders of magnitude higher than discussed here (see, e.g., Haensel et al. 2002 and references therein). The results may also differ significantly for superfluid neutron stars, because superfluidity drastically changes the reaction rates in dense matter and, hence, its kinetic properties, including the viscosity. It would also be instructive to consider other types of neutron star pulsations, primarily r-modes. They can be accompanied by the emission of gravitational waves (see, e.g., Andersson & Kokkotas 2001) which can, in principle, be registered by next generation gravitational detectors. We expect to continue the analysis of the evolution of pulsating neutron stars.

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APPENDIX: NON-EQUILIBRIUM DIRECT URCA PROCESS

If the non-equilibrium direct Urca process is allowed in a pulsating neutron star, it can also be described by the quantities $Q_{\text{non-eq}}$ and Q_{bulk} given by equations (12) and (31), respectively. These quantities are taken from Yakovlev et al. (2001) and are denoted here as $Q^{(\text{D})}_{\text{non-eq}}(y)$ and $Q^{(\text{D})}_{\text{bulk}}(y)$. The averaging over a pulsation period yields

$$\overline{Q}^{(\text{D})}_{\text{non-eq}} = Q^{(\text{D})}_{\text{eq}} \left(1 + \frac{1071\pi^2 y_0^2}{914} + \frac{945\pi^4 y_0^4}{3656} + \frac{105\pi^6 y_0^6}{7312}\right), \quad \text{(A1)}$$

$$\overline{Q}^{(\text{D})}_{\text{bulk}} = \frac{714\pi^2}{457} Q^{(\text{D})}_{\text{eq}} \left(\frac{y_0^2}{2} + \frac{15\pi^2 y_0^4}{68} + \frac{5\pi^4 y_0^6}{272}\right), \quad \text{(A2)}$$

where $Q^{(\text{D})}_{\text{eq}}$ is the neutrino emissivity of the direct Urca process and (as before) $y_0 = \delta\mu/(\pi^2\, k_{\text{B}}T)$. In a pulsating star with the allowed direct Urca process, $\overline{Q}^{(\text{D})}_{\text{non-eq}}$ and $\overline{Q}^{(\text{D})}_{\text{bulk}}$ should be included in the quantities $\overline{Q}_{\text{non-eq}}$ and $\overline{Q}_{\text{bulk}}$ given by equations (34) and (43), respectively. In the absence of nucleon superfluidity, the contribution of the direct Urca process into $\overline{Q}^{(\text{D})}_{\text{non-eq}}$ and $\overline{Q}^{(\text{D})}_{\text{bulk}}$ is five to seven orders of magnitude greater than that of the modified Urca process.

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