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Safety Functionals for Time Delay Systems

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Abstract— **This paper considers the stability and safety of nonlinear time delay systems. Utilizing a Lyapunov-Krasovskii functional we state and prove Lyapunov's theorem in its modern form and prove it with the help of the comparison lemma. Based on this we establish the notion of safety functional that allows us to ensure invariance of sets in the infinite-dimensional state space. The applicability of the results are also demonstrated using an illustrative example.**

I. Introduction

Time delays have been ubiquitous in many control systems, including, human balancing [1], neural networks [2] and connected vehicle systems [3]. There has been many efforts in establishing stability properties of time delay systems during the last couple of decades both on the time domain and on the frequency domain [4]–[6]. In particular, stability of systems with constant and time-varying delays was established with the help of Lyapunov-Krasovskii functionals [7], [8]. However, finding such functionals is typically hard and the results obtained are often conservative (which is generally a drawback of Lyapunov-based approaches).

In the context of systems without time delays, recent efforts on nonlinear control focused on safety by looking at the forward invariance of sets in state space. This is originated from the "modern" version of Lyapunov approach that focuses on mapping the dynamics to a one dimensional dynamical system via class $\mathcal{K}$ functions and utilizing the comparison lemma in the proofs [9]–[11]. Based on this approach the concept of barrier functions (or as we call here safety functions) has been established that allows one to ensure set invariance [11]. This concept has already been successfully applied to establish collision-free motion for multi-robot systems [12] and for automated and connected vehicles [13], [14]. However, when time delays are introduced into the dynamics they may still render the system unsafe even when safety is ensured for the non-delayed system. In this paper we focus on extending the concept of safety to time delay systems.

In [15] time delays were approximated so that safety functions were applicable. As such approximations may not always be feasible, here we consider no approximation and establish tools for set invariance in the infinite-dimensional state space. We start with recalling the results for non-delayed system and we highlight the connections between Lyapunov functions and safety functions. Then we state Lyapunov's theorem in its modern form by utilizing Lyapunov-Krasovskii functionals that map the states in the infinite-dimensional state space to scalars. We prove this theorem utilizing the comparison lemma. This proof naturally leads to the concept of safety functionals that can guarantee forward invariance of sets in state space. The safety conditions are summarized in another theorem that is also proven with the help of the comparison lemma. A simple example is used to illustrate the abstract concepts.

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II. Motivation

In this section we discuss the stability and safety of dynamical systems that can be described by ordinary differential equations (ODEs). In particular, we describe the stability of equilibria of autonomous dynamical systems using Lyapunov's approach presenting it in its modern form. This approach naturally leads to the notion of safety function that can be used to ensure forward invariance of a given domain in state space. We also show an illustrative example to demonstrate the practical use of these concepts. For simplicity we consider the initial conditions at $t = 0$ and this is kept in the rest of the paper.

Consider the dynamical system described by the ODE

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where the dot represent derivative with respect to time t, $x \in \mathbb{R}^n$, and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous function defined on an open and connected set within $\mathbb{R}^n$. Moreover, without loss of generality we assume $f(0) = 0$, that is, $x(t) \equiv x^* = 0$ for $t \geq 0$ is an equilibrium of (1).

In order to ensure the stability of the equilibrium one may use Lyapunov functions as stated by the following theorem:

Theorem 1. *Consider the continuously differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$ satisfying $V(0) = 0$. Then the trivial equilibrium $x^* = 0$ is asymptotically stable if*

$$\alpha_1(||x||) \leq \quad V(x) \quad \leq \alpha_2(||x||)\,, \tag{2}$$

$$\dot{V}(x) \quad \leq -\alpha_3(||x||)\,, \tag{3}$$

where $\dot{V}(x) = \frac{\partial V(x)}{\partial x} \cdot f(x)$ denotes the derivative along solutions of (1) and α_i, $i = 1, 2, 3$ are class $\mathcal{K}$ functions. Moreover, we have

$$||x(t)|| \leq \alpha_1^{-1}\big(\beta(\alpha_2(||x(0)||), t)\big), \tag{4}$$

for $t \geq 0$ where β is a class $\mathcal{KL}$ function and it is the solution of the initial value problem

$$\dot{y}(t) = -\alpha_3\big(\alpha_2^{-1}(y(t))\big), \qquad y(0) = V(x(0)). \tag{5}$$

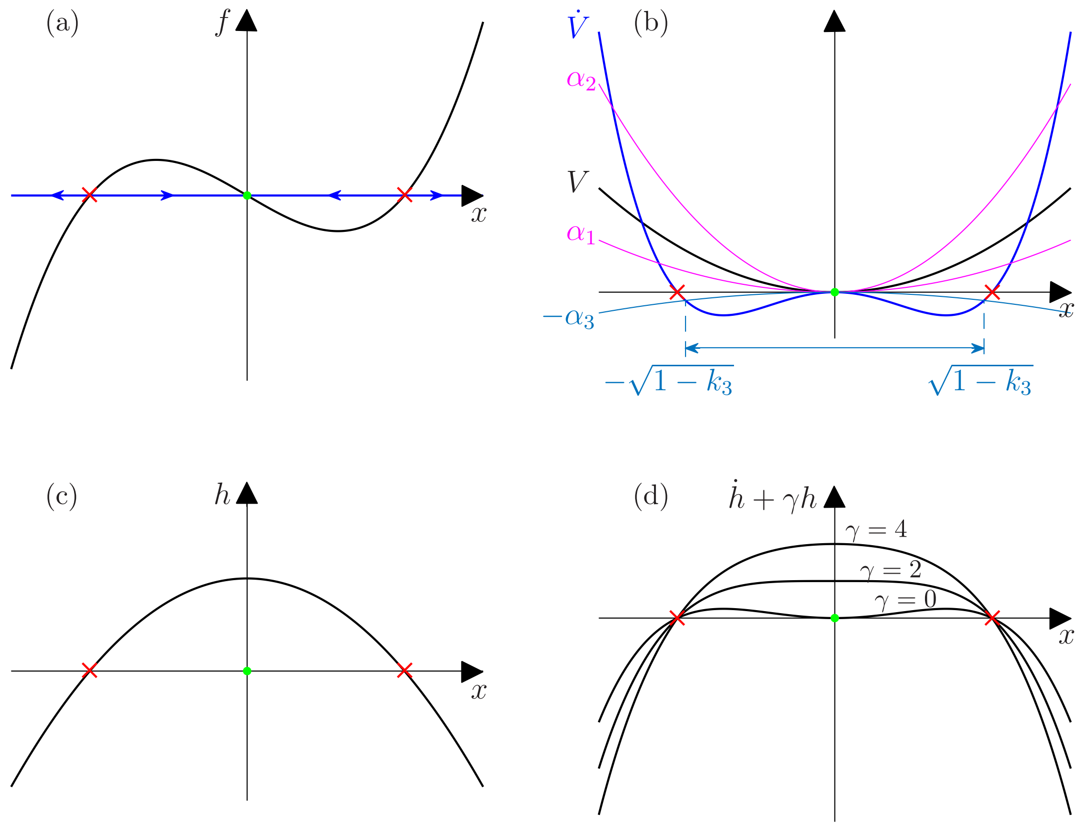

Fig. 1: (a) The nonlinear function f representing the right hand side in example (14). Stable and unstable equilibria are highlighted by green dots and red crosses, respectively, while the blue arrows indicate the flow between them. (b) The Lyapunov function V, its Lie derivative $\dot{V}$, and the class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$. The domain of attraction is highlighted. (c) The safety function h. (d) The function $\dot{h} + \gamma h$ appearing in the safety condition for different values of γ as indicated.

Proof: As α_2 is a class $\mathcal{K}$ from (2,3) we obtain

$$\dot{V}(x) \leq -\alpha_3\big(\alpha_2^{-1}(V(x))\big), \tag{6}$$

where $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function. Then using the comparison lemma [10] we obtain

$$V(x(t)) \leq \beta(V(x(0)), t), \tag{7}$$

for $t \geq 0$ where $y(t) = \beta(y(0), t)$ is the unique solution of the initial value problem (5). Since α_1 is a class $\mathcal{K}$ function from (2,7) we obtain

$$\begin{aligned} ||x(t)|| &\leq \alpha_1^{-1}(V(x(t))) \\ &\leq \alpha_1^{-1}\big(\beta(V(x(0)), t)\big) \\ &\leq \alpha_1^{-1}\big(\beta(\alpha_2(||x(0)||), t)\big), \end{aligned} \tag{8}$$

for $t \geq 0$. Since $\alpha_1^{-1} \circ \beta(\alpha_2(r), s)$ is a class $\mathcal{KL}$ function asymptotic stability is assured. ■

However, guaranteeing stability may not be required in many application but we often simply want to make sure that the system stays in given set in state space. This leads to the following construction.

Definition 1. *The set $S \subset \mathbb{R}^n$ is a super level set of a continuously differentiable function $h\colon \mathbb{R}^n \to \mathbb{R}$ if*

$$\begin{aligned} S &= \{x \in \mathbb{R}^n : h(x) \geq 0\}, \\ \partial S &= \{x \in \mathbb{R}^n : h(x) = 0\}, \\ \mathrm{Int}(S) &= \{x \in \mathbb{R}^n : h(x) > 0\}. \end{aligned} \tag{9}$$

We define *safety* of a system in terms of the set S being forward invariant under the dynamics (1). In particular, since we are assuming that f is Lipschitz continuous in (1), given the initial condition $x(0) = x_0$, there exists an interval $I(x_0) = [0, T(x_0)) \subset \mathbb{R}$ where the solution $x(t)$ exists. If solutions are forward complete we have $T(x_0) \to \infty$. Thus, we say that a set S is *forward invariant* if for all initial conditions $x(0) = x_0 \in S$, it follows that the solution $x(t) \in S$ for all $t \in I(x_0)$. This motivates viewing set invariance in the context of safety: the set S represents the safe configurations of the system, and the system is safe if it stays within these safe configurations.

This motivates the following Lyapunov view point of safety (first presented in [16], and studied in detail in [11]).

Theorem 2. *Given the set $S \subset \mathbb{R}^n$ that is a super level set of a continuously differentiable function $h\colon \mathbb{R}^n \to \mathbb{R}$, it is forward invariant if*

$$\dot{h}(x) \geq -\alpha(h(x)), \tag{10}$$

*where $\dot{h}(x) = \frac{\partial h(x)}{\partial x} \cdot f(x)$ is the derivative along solutions of (1) and α is a class $\mathcal{K}$ function. In this case S is called the **safety set** while h is called the **safety function**.*

Proof: To be able to use the comparison lemma [10] we set up the initial value problem

$$\dot{y}(t) = -\alpha(y(t)), \qquad y(0) = h(x(0)), \tag{11}$$

with the unique solution

$$y(t) = \beta(y(0), t) = \beta(h(x(0)), t), \tag{12}$$

for $t \geq 0$ where β is a class $\mathcal{KL}$ function. For $x_0 \in S$ and $x(t)$ denoting the solution of (1) with $x(0) = x_0$, we apply the comparison lemma for (10,11) and obtain

$$h(x(t)) \geq \beta(h(x(0)), t). \tag{13}$$

for $t \in I(x_0)$. This in turn implies that $h(x(t)) \geq 0$ for $t \in I(x_0)$, that is, S is forward invariant. ■

Note that safety functions are motivated by, and similar to, Lyapunov functions. Yet, there are key differences: since the objective is to create set invariance, rather than stabilize to a point, the positive definite assumption of Lyapunov functions (as encoded by (2)) is no longer needed. The reason is that, while we want to ensure invariance of S, we do not necessarily want to force invariance of sublevel sets—the end result is necessary and sufficient conditions for set invariance [11]. That being said, we can construct safety functions from Lyapunov functions. That is, one may construct a safety set as a set contained inside a contour of a Lyapunov function that corresponds to defining the safety function as $h(x) = c - V(x)$. For more details on barrier/safety functions of different types; see [11].

To motivate these concepts, consider the following example of a scalar nonlinear system:

$$\dot{x}(t) = -x(t) + x^3(t), \tag{14}$$

where $x \in \mathbb{R}$. The solution of this system can be written explicitly as

$$x(t) = \pm \frac{1}{\sqrt{\left(\frac{1}{x^2(0)} - 1\right)\mathrm{e}^{2t} + 1}}, \tag{15}$$

where the $+$ $(-)$ shall be applied when $x(0) > 0$ $(x(0) < 0)$. The system has three equilibria of the form $x(t) = x^*$ for $t \geq 0$. Apart from the trivial stable equilibrium $x^* = 0$ there exist two unstable equilibria $x^* = 1$ and $x^* = -1$. In Fig. 1(a) the right hand side of (14) is plotted together with the stable and unstable equilibria. One may conclude from this figure that the region of attraction of the trivial equilibrium is given by $x \in (-1, 1)$.

When applying (2) in Theorem 1 one may choose the Lyapunov function $V(x) = x^2/2$ that is bounded by the class $\mathcal{K}$ functions $\alpha_1(||x||) = k_1 x^2$, $k_1 \leq 1/2$ and $\alpha_2(||x||) = k_2 x^2$, $k_2 \geq 1/2$. In this case the Lie derivative of the Lyapunov function becomes $\dot{V}(x) = -x^2 + x^4$. Thus, choosing $\alpha_3(||x||) = k_3 x^2$ in (3) one can create a lower bound for the Lie derivative on the interval $x \in (-\sqrt{1-k_3}, \sqrt{1-k_3})$ as highlighted in Fig. 1(b). Finally, (4) results in $||x(t)|| \leq \sqrt{\frac{k_2}{k_1}} ||x(0)|| \mathrm{e}^{-\frac{k_3}{2k_2} t}$ within this region of attraction. We remark that considering $k_1 = k_2 = 1/2$ we have $||x(t)|| \leq ||x(0)|| \mathrm{e}^{-k_3 t}$.

The Lyapunov results indicate that the interval $x \in (-1, 1)$ is forward invariant and this can be proven by using the safety function $h(x) = 1/2 - x^2/2$ as shown Fig. 1(c). Choosing the class $\mathcal{K}$ function $\alpha(h) = \gamma h$ with $\gamma > 0$ in (10) we obtain the condition $\dot{h}(x) + \gamma h(x) = -x^4 + (1 - \gamma/2)x^2 + \gamma/2 \geq 0$ that indeed holds on $x \in (-1, 1)$ as illustrated in Fig. 1(d).

III. Stability and Safety for Time Delay System

In this section we set up the state space representation of autonomous time delay systems for a single delay. Indeed the setup may also be generalized for multiple delays and distributed delays. Then by defining the Lyapunov-Krasowsii functional we state Lyapunov's theorem in its modern form and prove it with the help of the comparison lemma. This will lay the basis for the safety functionals and the corresponding theorem given at the end of the section.

We consider dynamical systems in the form given by the delay differential equation (DDE)

$$\dot{x}(t) = f(x(t), x(t-\tau)), \tag{16}$$

where $x \in \mathbb{R}^n$, and $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous function defined on an open and connected set within $\mathbb{R}^n \times \mathbb{R}^n$. Again, without loss of generality we assume $f(0, 0) = 0$, that is, $x(t) \equiv x^* = 0$ for $t \geq 0$ is an equilibrium of (16). Let us define $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$ which is a function in the Banach space $\mathcal{B} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ of continuous functions mapping the interval $[-\tau, 0]$ to $\mathbb{R}^n$. Notice that the initial condition for (16) is not simply $x(0)$ but rather the whole function $x_0(\theta) = x(\theta)$, $\theta \in [-\tau, 0]$ as illustrated by the green section in Fig. 2.

System (16) can be written as a functional differential equation

$$\dot{x}(t) = \mathcal{F}(x_t), \tag{17}$$

where the functional $\mathcal{F}\colon \mathcal{B} \to \mathbb{R}^n$ is defined by

$$\mathcal{F}(\phi) = f\big(\mathcal{G}_0(\phi), \mathcal{G}_{-\tau}(\phi)\big), \tag{18}$$

where

$$\mathcal{G}_\sigma(\phi) = \int_{-\tau}^{0} \phi(\theta)\delta(\theta - \sigma)\mathrm{d}\theta = \phi(\sigma), \tag{19}$$

and δ stands for the Dirac delta.

We remark that using $\mathcal{B}$ as state space one may also write (16) as an operator differential equation

$$\dot{x}_t = \mathcal{Q}(x_t), \tag{20}$$

where the operator $\mathcal{Q}\colon \mathcal{B} \to \mathcal{B}$ is given by

$$\mathcal{Q}(\phi)(\theta) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}\theta}\phi(\theta), & \text{if} \quad \theta \in [-\tau, 0), \\ f(\phi(0), \phi(-\tau)), & \text{if} \quad \theta = 0. \end{cases} \tag{21}$$

The time evolution of x_t in the infinite-dimensional state space $\mathcal{B} = \mathcal{C}([-\tau, 0], \mathbb{R})$ is illustrated in Fig. 2 where the headpoint $x_t(0) = x(t)$ is also highlighted. Note that the equilibria of (20) is given by $x_t(\theta) = x(t + \theta) = x^*$,

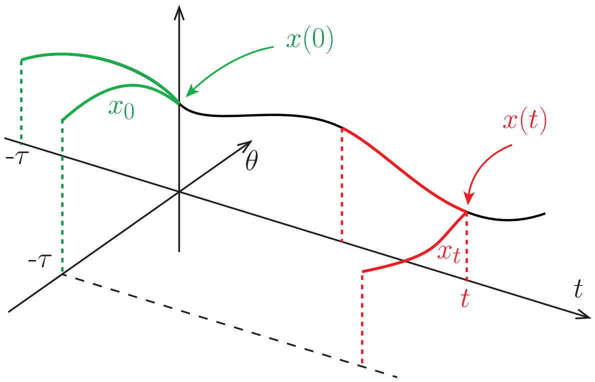

Fig. 2: Time evolution in the infinite-dimensional state space $\mathcal{B} = \mathcal{C}([-\tau, 0], \mathbb{R})$. The green section highlights the initial condition $x_0(\theta) = x(\theta)$, $\theta \in [-\tau, 0]$ with the headpoint $x_0(0) = x(0)$ while the red section highlights the state $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$ with the headpoint $x_t(0) = x(t)$.

$\theta \in [-\tau, 0]$, $t \geq 0$. See [4]–[6] for more details about the mathematical background of time delay systems.

The above description makes it evident that Lyapunov functions (that map $\mathbb{R}^n$ to $\mathbb{R}$) are not adequate to establish stability in the infinite-dimensional state space $\mathcal{B}$. Instead, one may construct a so-called Lyapunov-Krasovskii functionals that map $\mathcal{B}$ to $\mathbb{R}$. This is summarized in the following theorem, which extends modern version of Lyapunov theory to time delay systems.

Theorem 3. *Consider the continuously differentiable functional $\mathcal{V}\colon \mathcal{B} \to \mathbb{R}$ satisfying $\mathcal{V}(0) = 0$. Then the trivial equilibrium $x^* = 0$ is asymptotically stable if*

$$\alpha_1(||\phi||_{\rm s}) \leq \quad \mathcal{V}(\phi) \quad \leq \alpha_2(||\phi||_{\rm s}), \tag{22}$$

$$\dot{\mathcal{V}}(\phi) \quad \leq -\alpha_3(||\phi(0)||), \tag{23}$$

where $\dot{\mathcal{V}}$ is the derivative along solutions of (16) *and α_i, $i = 1, 2, 3$, are class $\mathcal{K}$ functions. Moreover,*

$$||x_t||_{\rm s} \leq \alpha_1^{-1}\big(\beta(\alpha_2(||x_0||_{\rm s}), t)\big), \tag{24}$$

for $t \geq 0$ where β is a class $\mathcal{KL}$ function and it is the solution of the initial value problem

$$\dot{y}(t) = -\alpha_3\big(\alpha_2^{-1}(y(t))\big), \qquad y(0) = \mathcal{V}(x_0). \tag{25}$$

Note that $\phi(0) \in \mathbb{R}^n$ and $||\phi(0)||$ denotes the norm in $\mathbb{R}^n$. On the other hand $\phi \in \mathcal{B}$ and $||\phi||_{\rm s} = \sup_{\theta\in[-\tau,0]} ||\phi(\theta)||$ is a sup norm. Indeed, we have $||\phi(0)|| \leq ||\phi||_{\rm s}$. We also remark that in [4] the left inequality $\alpha_1(||\phi(0)||) \leq$ is used in (22). Here we strengthened this condition by using $\alpha_1(||\phi||_{\rm s}) \leq$ which allows us derive (24) as shown in the proof below.

Proof: As α_2 is a class $\mathcal{K}$ from (22) we obtain

$$\alpha_2^{-1}(\mathcal{V}(\phi)) \geq ||\phi||_{\rm s} \geq ||\phi(0)||. \tag{26}$$

Putting this into (23) we get

$$\dot{\mathcal{V}}(\phi) \leq -\alpha_3\big(\alpha_2^{-1}(\mathcal{V}(\phi))\big), \tag{27}$$

where $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function.

In order to prepare for the comparison lemma we consider the initial value problem

$$\dot{y}(t) = -\alpha_3(\alpha_2^{-1}(y(t))), \qquad y(0) = \mathcal{V}(x_0), \tag{28}$$

which has the unique solution

$$y(t) = \beta(y(0), t) = \beta(\mathcal{V}(x_0), t), \tag{29}$$

for $t \geq 0$. Then using $\phi = x_t$ in (27) and applying the comparison lemma we obtain

$$\mathcal{V}(x_t) \leq \beta(\mathcal{V}(x_0), t), \tag{30}$$

for $t \geq 0$.

As α_1 is a class $\mathcal{K}$ function from (22) we also have

$$||\phi||_{\rm s} \leq \alpha_1^{-1}(\mathcal{V}(\phi)). \tag{31}$$

Substituting $\phi = x_t$ and using (30) we obtain

$$||x_t||_{\rm s} \leq \alpha_1^{-1}\big(\beta(\mathcal{V}(x_0), t)\big) \leq \alpha_1^{-1}\big(\beta(\alpha_2(||x_0||_{\rm s}), t)\big), \tag{32}$$

for $t \geq 0$. Note that in the last step we used (22) again. Since $\alpha_1^{-1} \circ \beta(\alpha_2(r), s)$ is a class $\mathcal{KL}$ function we can conclude asymptotic stability. ■

In order to ensure safety of time delay systems we need to ensure forward invariance of sets in the infinite-dimensional state space $\mathcal{B}$. This motivates the following definition

Definition 2. *The set $\mathcal{S} \subset \mathcal{B}$ is a super level set of a continuously differentiable functional $\mathcal{H}\colon \mathcal{B} \to \mathbb{R}$ if*

$$\begin{aligned} \mathcal{S} &= \{\phi \in \mathcal{B} : \mathcal{H}(\phi) \geq 0\}, \\ \partial\mathcal{S} &= \{\phi \in \mathcal{B} : \mathcal{H}(\phi) = 0\}, \\ \mathrm{Int}(\mathcal{S}) &= \{\phi \in \mathcal{B} : \mathcal{H}(\phi) > 0\}. \end{aligned} \tag{33}$$

In the context of these sets, the following theorem ensures forward invariance, i.e., safety. Thus, the following result generalizes safety functions to time delay systems.

Theorem 4. *Given the set $\mathcal{S} \subset \mathcal{B}$ that is a super level set of a continuously differentiable functional $\mathcal{H}\colon \mathcal{B} \to \mathbb{R}$, it is forward invariant if*

$$\dot{\mathcal{H}}(x_t) \geq -\alpha(\mathcal{H}(x_t)), \tag{34}$$

where α is a class $\mathcal{K}$ function. In this case $\mathcal{S}$ is called the **safety set** *while $\mathcal{H}$ is called the* **safety functional**.

Proof: Here we set up the initial value problem

$$\dot{y}(t) = -\alpha(y(t)), \qquad y(0) = \mathcal{H}(x_0), \tag{35}$$

when preparing for the comparison lemma. This has the unique solution

$$y(t) = \beta(y(0), t) = \beta(\mathcal{H}(x_0), t), \tag{36}$$

for $t \geq 0$ where β is a class $\mathcal{KL}$ function. Then, applying the comparison lemma for (34,35) we obtain

$$\mathcal{H}(x_t) \geq \beta(\mathcal{H}(x_0), t), \tag{37}$$

for $0 \leq t \leq T(x_0)$. This leads to $\mathcal{H}(x_t) \geq 0$ for $0 \leq t \leq T(x_0)$, that is, $\mathcal{S}$ is forward invariant. ■

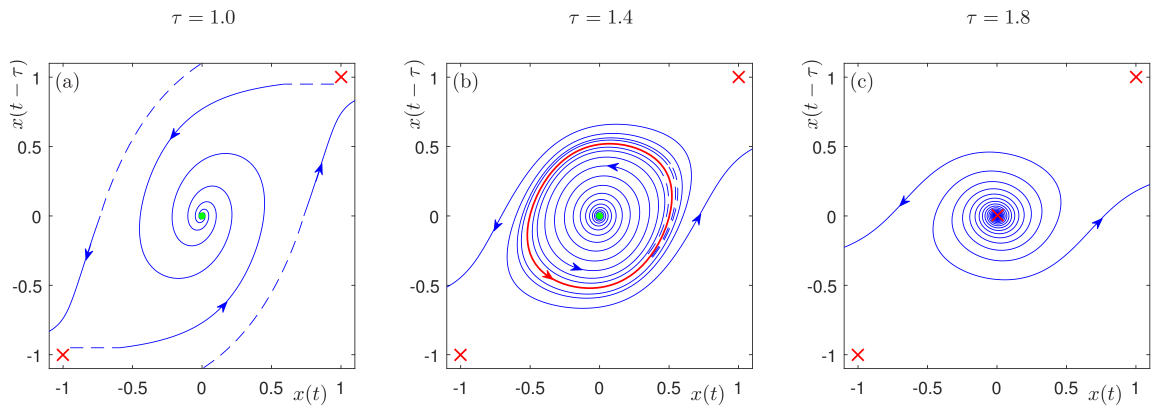

Fig. 3: State space plots for the example (38) for different values of the delay as indicated above. Stable and unstable equilibria are denoted by green dots and red crosses, respectively. In panel (b) the red circle indicates an unstable limit cycle. The dashed section of the blue trajectories correspond to the initial conditions.

Here, as discussed after Definition 1, $T(x_0)$ denotes time until the solution exists. This may be finite as we are lower bounding the $\dot{\mathcal{H}}$ dynamics though for some initial conditions $T(x_0) \to \infty$ may occur.

IV. Illustrative Example

In order to illustrate the applicability of the above theorems we return to the example (14). More precisely, we consider the nonlinear time delay system

$$\dot{x}(t) = -x(t-\tau) + x^3(t), \tag{38}$$

where $x \in \mathbb{R}$. This system has three equilibria of the form $x_t(\theta) = x(t+\theta) = x^*$, $\theta \in [-\tau, 0]$, $t \geq 0$. Similar to the non-delayed case, these are still at $x^* = 0$, $x^* = 1$ and $x^* = -1$. Using linear stability analysis one may show that the latter two equilibria are unstable for any value of τ while the trivial equilibrium is linearly stable for $\tau < \pi/2$ and unstable for $\tau > \pi/2$. It can be shown using bifurcation analysis that this equilibrium looses stability via a subritical Hopf bifurcation resulting in a unstable limit cycle for the delay range $1.162 < \tau < \pi/2$. In this parameter range the region of attraction of the trivial equilibrium is given by the stable manifold of the unstable limit cycle. Indeed, for $\tau < 1.162$ the region of attraction is bounded by the stable manifolds of the equilibria $x^* = 1$ and $x^* = -1$ while for $\tau > \pi/2$ the region of attraction disappears. The simulations in Fig. 3 demonstrate these three qualitatively different behaviors for different values of τ.

Our goal here is to find a safety functional and show that a safety set can be constructed about the trivial equilibrium and safety can be ensured if the delay τ is small enough. We follow the methodology given in Sec. II and start with the Lyapunov functional

$$\mathcal{V}(\phi) = \frac{1}{2}\phi^2(0) + \int_{-\tau}^{0} \phi^2(\theta) w(\theta) \mathrm{d}\theta, \tag{39}$$

where the density function satisfies $w(\theta) \geq 0$. Then we define the safety functional as

$$\mathcal{H}(\phi) = c - \mathcal{V}(\phi). \tag{40}$$

In the infinite-dimensional state space $\mathcal{B} = \mathcal{C}([-\tau, 0], \mathbb{R})$ the condition $\mathcal{H}(\phi) \geq 0$ defines an ellipsoid. Notice that for $c < 1/2$ this ellipsoid is contained by $||\phi(0)|| < 1$. The forward invariance of this ellipsoid is given by the safety condition (34).

Let us choose the class $\mathcal{K}$ function in (34) to be $\alpha(\mathcal{H}) = \gamma\mathcal{H}$ with $\gamma > 0$. Then we obtain

$$\begin{aligned} &\dot{\mathcal{H}}(x_t) + \gamma\mathcal{H}(x_t) \\ &= x(t)x(t-\tau) - x^4(t) - w(0)x^2(t) + w(-\tau)x^2(t-\tau) \\ &\quad + \int_{-\tau}^{0} x^2(t+\theta)\frac{\mathrm{d}w(\theta)}{\mathrm{d}\theta}\mathrm{d}\theta \\ &\quad + \gamma\left(c - \frac{1}{2}x^2(t) - \int_{-\tau}^{0} x^2(t+\theta)w(\theta)\mathrm{d}\theta\right) \geq 0, \end{aligned} \tag{41}$$

where we used (38). In order to be able to eliminate the integrals we select the density function $w(\theta) = A\mathrm{e}^{at}$ with $A > 0$ and $a = \gamma > 0$ that leads to

$$\begin{aligned} &x(t)x(t-\tau) - x^4(t) - Ax^2(t) + A\mathrm{e}^{-\gamma\tau}x^2(t-\tau) \\ &\quad + \gamma\left(c - \frac{1}{2}x^2(t)\right) \geq 0. \end{aligned} \tag{42}$$

In Fig. 4 condition (42) is represented in the $(x(t), x(t-\tau))$-plane for different values of the delay τ and the parameters A and γ. Recall that we selected $w(\theta) = A\mathrm{e}^{\gamma\theta}$, that is, A represents the weight of the integral in the safety functional (39,40) while γ represents how fast that integral decays. Here we also considered $c = 1/4 < 1/2$. The domain between a pair of curves of the same color corresponds to where condition (42) holds. For each parameter combination presented there exists a domain around the origin where the condition satisfied. However, the condition needs to be satisfied on the whole ellipsoid $\mathcal{H}(\phi) \geq 0$ in the infinite-dimensional state space which we found to be challenging to compute. Still, the state space plots in Fig. 3 imply that this can only be shown for smaller values of the delay.

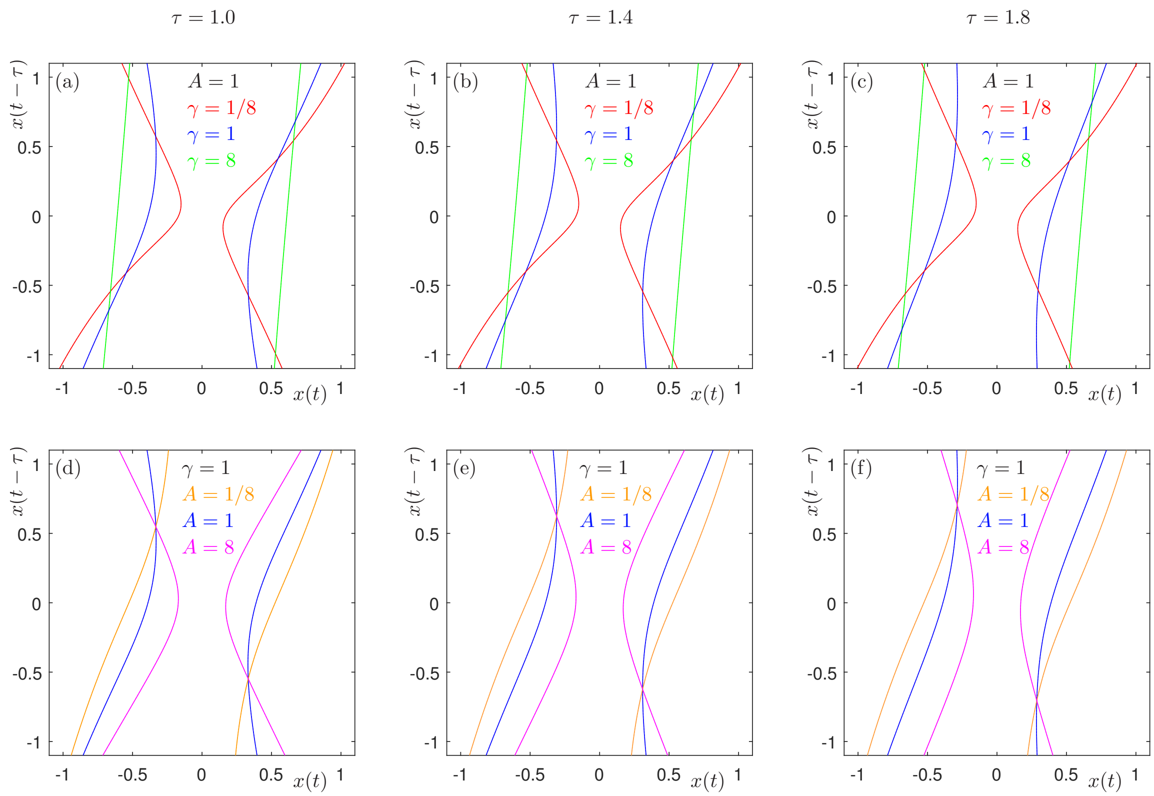

Fig. 4: Graphical representation of condition (42) for $c = 1/4$ and different values of τ, A, and γ as indicated. In each case the condition holds between the curves of a given color.

V. Conclusion

In this paper we discussed stability and safety (framed as set invariance) in time delay systems. In particular, for autonomous dynamical systems we proved Lyapunov's theorem the "duel" safety theorem with the help of the comparison lemma while utilizing class $\mathcal{K}$ and class $\mathcal{KL}$ functions, i.e., we presented the "modern" formulation of these theorems. This led to the definition of the safety functional that can be used to ensure safety in the presence of time delays. Open questions include the the explicit computation of the safety domain and the optimization of parameters in the safety functional that ensure the largest safety domain. Also, extending these concept for control design, e.g., mirroring control barrier functions, is left for future research.

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