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Cluster synchronization, switching and spatiotemporal coding in a phase oscillator network

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A network of five globally-coupled identical phase oscillators is considered. Cluster states consisting of two synchronized pairs of oscillators and one singleton are investigated. Forcing the system with non-uniform constant inputs results in regular switches between cluster states. The resultant cyclic sequences of switches (spatiotemporal codes) are studied for different initial conditions and input configurations. Implications on information coding in neural systems are briefly discussed.

1 Cluster states in a phase oscillator model

We consider the phase oscillator model introduced in [1] for five identical oscillators with all-to-all coupling. The governing ordinary differential equations are given by

$$\dot{\theta_n}(t) = \omega + \frac{1}{5}\sum_{m=1}^{5} g\big(\theta_n(t) - \theta_m(t)\big) + p\, I_n\,, \quad n = 1, \ldots, 5\,, \tag{1}$$

where $\theta_n(t) \in [0, 2\pi)$ is the phase of the n-th oscillator, the dot represents differentiation with respect to the time t, ω is the common natural frequency of oscillators and the coupling function is defined as $g(\varphi) = -\sin(\varphi+\alpha) + r\sin(2\varphi+\beta)$. In this paper we use $\omega = 1.0$, $r = 0.2$, $\alpha = 1.8$ and $\beta = -2.0$ but the dynamics explained below are robust to small changes in these parameters [1]. The term $p\, I_n$ represents the constant external stimulus received by the n-th oscillator. The input configuration $[I_1\, I_2\, I_3\, I_4\, I_5\,]$ is a permutation of $[\,1\,2\,3\,4\,5\,]$, so each oscillator receives a different constant stimulus of the order of the input magnitude $p \ll \omega$. (Note that there exist $5! = 120$ different input configurations.) One may consider an input as a detuning to the natural frequency ω leading to the altered frequencies $\omega_n = \omega + p\, I_n$. Note that the right-hand side of eq. (1) only depends on the phase differences, so it is sufficient to examine these to determine the behavior of the system.

In was shown in [1] that for $p = 0$ there exists a partially synchronized cluster state $[\,\theta_1(t)\,\theta_2(t)\,\theta_3(t)\,\theta_4(t)\,\theta_5(t)\,] = \Omega\, t\,[\,1\,1\,1\,1\,1\,] + [\,y\,y\,z\,x\,x\,]$ with constant frequency Ω and phases x, y and z. For the above parameters we have $\Omega \simeq 0.85$, $x - z \simeq 1.10$ and $y - z \simeq -1.82$. (Only the phase differences are known which is sufficient). Furthermore, there are additional cluster states obtained by permuting the components of $[\,y\,y\,z\,x\,x\,]$. All the $5!/(2!\,1!\,2!) = 30$ cluster states are listed in Table 1 (where the term $\Omega\, t\,[\,1\,1\,1\,1\,1\,]$ is not spelled out).

$[\,y\,y\,z\,x\,x\,] = s_1$	$[\,x\,y\,y\,z\,x\,] = s_7$	$[\,x\,x\,y\,y\,z\,] = s_{13}$	$[\,z\,x\,x\,y\,y\,] = s_{19}$	$[\,y\,z\,x\,x\,y\,] = s_{25}$
$[\,x\,x\,z\,y\,y\,] = s_2$	$[\,y\,x\,x\,z\,y\,] = s_8$	$[\,y\,y\,x\,x\,z\,] = s_{14}$	$[\,z\,y\,y\,x\,x\,] = s_{20}$	$[\,x\,z\,y\,y\,x\,] = s_{26}$
$[\,y\,x\,z\,y\,x\,] = s_3$	$[\,x\,y\,x\,z\,y\,] = s_9$	$[\,y\,x\,y\,x\,z\,] = s_{15}$	$[\,z\,y\,x\,y\,x\,] = s_{21}$	$[\,x\,z\,y\,x\,y\,] = s_{27}$
$[\,x\,y\,z\,x\,y\,] = s_4$	$[\,y\,x\,y\,z\,x\,] = s_{10}$	$[\,x\,y\,x\,y\,z\,] = s_{16}$	$[\,z\,x\,y\,x\,y\,] = s_{22}$	$[\,y\,z\,x\,y\,x\,] = s_{28}$
$[\,y\,x\,z\,x\,y\,] = s_5$	$[\,y\,y\,x\,z\,x\,] = s_{11}$	$[\,x\,y\,y\,x\,z\,] = s_{17}$	$[\,z\,x\,y\,y\,x\,] = s_{23}$	$[\,x\,z\,x\,y\,y\,] = s_{29}$
$[\,x\,y\,z\,y\,x\,] = s_2$	$[\,x\,x\,y\,z\,y\,] = s_{12}$	$[\,y\,x\,x\,y\,z\,] = s_{18}$	$[\,z\,y\,x\,x\,y\,] = s_{24}$	$[\,y\,z\,y\,x\,x\,] = s_{30}$

Table 1 List of the saddle cluster states for eq. (1). (Each column may be generated from another by cyclic permutation of the components.)

2 Switching between cluster states and spatiotemporal codes

Stability investigations in [1] showed that the above cluster states are saddle type and they are connected by their unstable manifolds to form a (stable) heteroclinic network. This network governs the appearing switching dynamics between cluster states (also called the 'winnerless competition'), and it can be represented by a graph where the cluster states are represented by nodes and the switches are represented by directed edges. Each node has two incoming and two outgoing edges. For $p = 0$ the time interval between switches increases exponentially (oscillations slow down). For $p > 0$ there exists a characteristic switching time $T_{\rm s} \sim -\ln p$ and the heteroclinic network reduces to limit cycles which correspond to cyclic paths along the network. Which limit cycle is approached depends on the initial condition as shown in state space in Fig. 1(a1,b1) for the input $10^{-5}[\,1\,2\,3\,4\,5\,]$. The corresponding graphs of cyclic paths are displayed in Fig. 1(a2,b2). Consequently, the information

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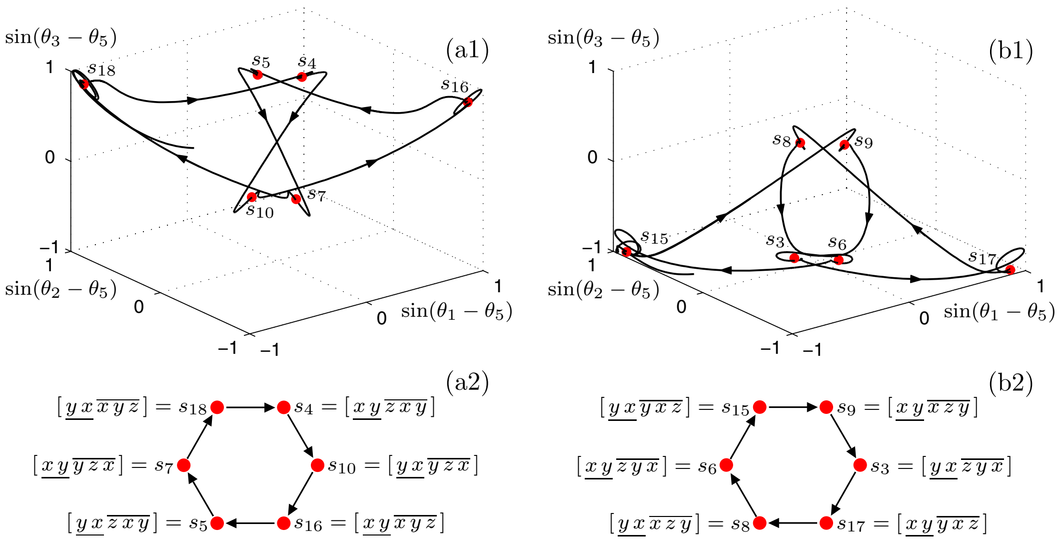

Fig. 1 Cyclic sequences of switches between cluster states in state space for two different initial conditions (a1,b1) and the corresponding graphs (a2,b2). The input configuration is $[\,I_1\,I_2\,I_3\,I_4\,I_5\,] = [\,1\,2\,3\,4\,5\,]$ and the input magnitude is $p = 10^{-5}$. In panels (a2,b2), the underlined oscillators (receiving stimuli $p\,[\,1\,2\,]$) swap their phases x and y at each switch while the overlined oscillators (receiving stimuli $p\,[\,3\,4\,5\,]$) cyclically permute their phases x, y and z.

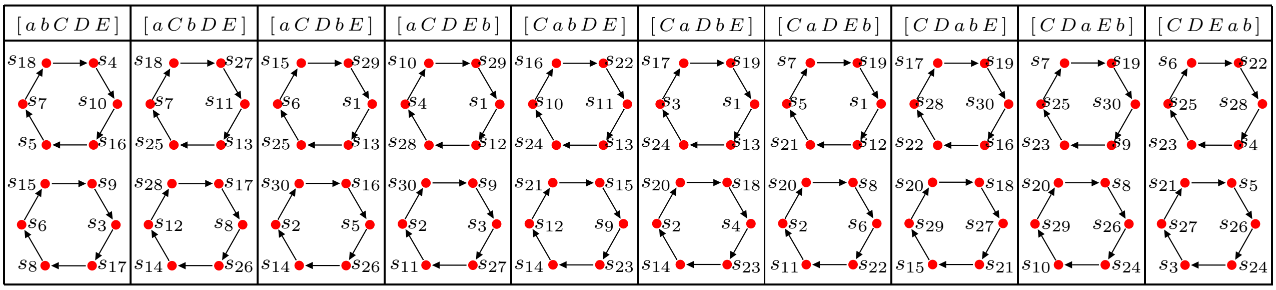

Fig. 2 The full code table with constant inputs and spatiotemporal outputs. In the input configuration, the pair $[\,a\,b\,]$ represents any of the two permutations of $[\,1\,2\,]$ while the triplet $[\,C\,D\,E\,]$ represents any of the six permutations of $[\,3\,4\,5\,]$.

about the input configuration $[\,1\,2\,3\,4\,5\,]$ is encoded into the appearing limit cycles/cyclic paths which can be considered as spatiotemporal codes. (Note that long transients, which include switches between other cluster states, are also possible but are not discussed here.)

It was shown in [1] that the cluster with phase x is the 'unstable' one. The state reached after a switch is determined by which oscillator with phase x receives the larger stimulus. The phase of this oscillator changes to z, the phase of the other 'x oscillator' changes to y and the phases of the 'z and y oscillators' change to y and x, respectively; see Fig. 1(a2,b2). Notice that along the cyclic paths in Fig. 1(a2,b2) the underlined oscillators (receiving stimuli $p\,[\,1\,2\,]$) swap their phases x and y at each switch while the overlined oscillators (receiving stimuli $p\,[\,3\,4\,5\,]$) cyclically permute their phases x, y and z. In fact, the same spatiotemporal codes are obtained for all input configurations with permutations within the pair $[\,1\,2\,]$ and within the triplet $[3\,4\,5]$ as shown by the first column of the code table in Fig. 2. (There are $2!\,3! = 12$ such cases). However, permutations which mix elements of the pair and the triplet results in different spatiotemporal codes; see Fig. 2. Thus the 120 input configurations are divided into $120/12 = 10$ groups where each group results in a different pair of spatiotemporal codes.

In insect antennal lobe systems, odour information is coded into spatiotemporal codes, that is both neural identity and interneuronal timing are used for coding [2]. This allows the system to specify a great number of different chemical mixtures. Our work shows dynamics of a simple system that can robustly give an immense variety of spatiotemporal codes.

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