Math 454 section 001, Exam 1 : February 13, 2014

Name : ______________________________ **UM ID #:**____________________

Instructions and notes

1. Turn off and put away your cell phone and other electronic devices. Please ensure that they will be **silent** for the duration of the exam.
2. You may use one page of notes (one side of one piece of letter size paper, 8.5 in $\times$ 11 in).
3. The use of calculators or any other electronic aids are not permitted on this exam.
4. Write your solutions clearly; no credit will be given for illegible solutions. Please clearly mark your ultimate result for each problem (i.e., by drawing a box or a circle around your answer).
5. This exam has 5 problems spread across 6 double-sided pages.
6. If you need more room to write than the space provided by the exam, write your name on **each** additional sheet of paper you turn in.

Academic integrity statement

Please copy the following statement, and sign below your writing : "I pledge that I have neither given nor received any unauthorized assistance regarding this exam."

signature : __________________________

possibly useful formulae :

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad \text{(Cartesian)}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}, \quad \text{(cylindrical)}$$

$$= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}, \quad \text{(spherical)}$$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & n = m = 0 \\ \frac{L}{2} & m = n \neq 0 \end{cases}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B, \quad \cos A\cos B = \tfrac{1}{2}\big(\cos(A-B) + \cos(A+B)\big)$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B, \quad \sin A\sin B = \tfrac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$$

$$\cosh(A \pm B) = \cosh A\cosh B \pm \sinh A\sinh B, \quad \cosh A\cosh B = \tfrac{1}{2}\big(\cosh(A+B) + \cosh(A-B)\big)$$

$$\sinh(A \pm B) = \sinh A\cosh B \pm \cosh A\sinh B, \quad \sinh A\sinh B = \tfrac{1}{2}\big(\cosh(A+B) - \cosh(A-B)\big)$$

heat equation :

K_0 = thermal conductivity, c = specific heat, ρ = mass density

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{(1-d)}, \quad \frac{\partial u}{\partial t} = k\nabla^2 u \quad \text{(2-d \& 3-d)}, \quad k = \frac{K_0}{c\rho} = \text{const.}$$

Problem	points	score
1.	10	
2.	15	
3.	20	
4.	15	
5.	20	
$\sum$	80	

1. **True / False.** Give a reason to justify your answer.
(10 points : 5 parts, 2 points each)

(a) It is possible to find coefficients b_n, $n = 1, \ldots, \infty$ such that $\cos x = \sum_{n=1}^{\infty} b_n \sin nx$ for all $x \in (-\pi, \pi)$.

(b) Every periodic function with period $2L$ is equal to its Fourier series expansion on $[-L, L]$.

(c) The function $f(x) = x^{1/5}$ is piecewise smooth on the interval $(-1, 1)$.

(d) Consider a metal rod of length L with cross-sectional area A (figure 1); if $u(x,t)$ represents the temperature in the rod at position $x \in [0,L]$ and time $t > 0$, then the quantity $\int_0^L c(x)\rho(x)u(x,t)A\,dx$ represents the total thermal energy within the rod at time t.

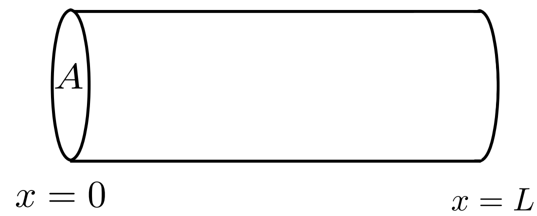

Figure 1.

(e) Suppose that $f(x)$ is continuous and that $f'(x)$ is piecewise continuous. Then the Fourier series of $f(x)$ can be differentiated term-by-term.

2. (15 points) Consider the function $f(x) = e^{-3x}$ on $0 < x < L$.

   (a) (3 pts) Define $f_{\text{odd}}(x)$, the odd extension of $f(x)$ on $-L < x < L$.

   (b) (3 pts) Define $f_{\text{even}}(x)$, the even extension of $f(x)$ on $-L < x < L$.

   (c) (3 pts) Let $\tilde{f}_c(x)$ denote the Fourier cosine series of $f(x)$. What is $\tilde{f}_c(0)$?

   (d) (3 pts) Let $\tilde{f}_s(x)$ denote the Fourier sine series of $f(x)$. What is $\tilde{f}_s(0)$?

   (e) (3 pts) Find b_n, the coefficients of the Fourier sine series of $f(x)$.

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3. (20 points) Consider the equilibrium temperature distribution of a uniform one-dimensional rod with sources $Q = K_0 x$ of thermal energy, subject to the boundary conditions $u(0) = u(L) = 0$.

   (a) (5 pts) Determine the heat energy generated per unit time inside the entire rod.

   (b) (10 pts) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and $x = L$.

   (c) (5 pts) What relationship should exist between the answers to parts (a) and (b)?

(workspace)

4. (15 points) Determine the equilibrium temperature distribution inside a circular annulus ($R_1 \leq r \leq R_2$), if the outer radius is at constant temperature T_2 and the inner radius is at constant temperature T_1. The domain is illustrated in figure 2.

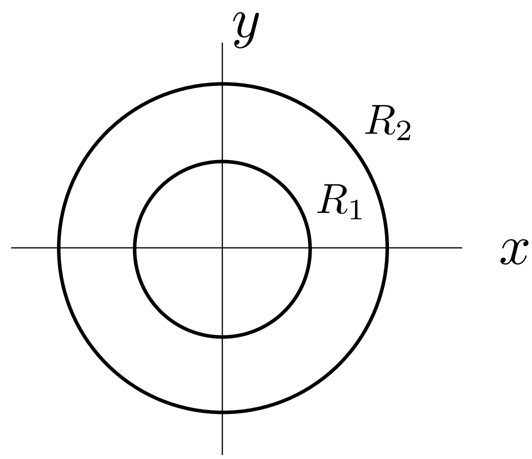

Figure 2.

(workspace)

5. (20 points) Use separation of variables or the method of eigenfunction expansion to solve the Laplace equation $\nabla^2 u = 0$ on the unit square, $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$, subject to the boundary conditions

$$\begin{aligned} u(0, y) &= 1, \qquad & u(1, y) &= 2\sin \pi y \\ u(x, 0) &= 0 \qquad & u(x, 1) &= 0. \end{aligned}$$

(workspace)