Math 471 Review Sheet for Final Exam Fall 2013

The final exam will cover the entire course. You may use two sheets of notes (e.g. two sides of one page, i.e. a total of $187\,\text{in}^2 = 2 \times 8.5\,\text{in} \times 11\,\text{in}$). You may use a non-programmable calculator to do arithmetic, but to receive full credit you must show all intermediate steps. Exam booklets will be provided.

final exam : Friday, 12/13, 10:30am-12:30pm, 1123 LBME

notes :

You will need a calculator that can compute trigonometric functions, logarithms, and exponentials. Graphing calculators are fine, but not programmable versions like the TI-92.

1. True or False? Give a reason to justify your answer.

   (a) If two floating point numbers with n significant digits are added, then the result also has n significant digits.

   (b) $D_0 f(x) = f'(x) + O(h^2)$

   (c) If $Ax = 0$, then $A = 0$ or $x = 0$.

   (d) If A is invertible, then $||A||^{-1} \leq ||A^{-1}||$.

   (e) $\rho(B) \leq ||B||_\infty$ for any matrix B

   (f) The spectral radius of a matrix satisfies the properties required to be a matrix norm.

   (g) If the pivots arising in Gaussian elimination are nonzero, then A is invertible.

   (h) In solving an $n \times n$ system of linear equations by Gaussian elimination, if n increases by a factor of 5, then the operation count increases by a factor of approximately 25.

   (i) In computing the solution of a linear system $Ax = b$, if the residual norm $||r||$ is small, then the error norm $||e||$ is also small.

   (j) In solving a linear system $Ax = b$ by an iterative method $x_{k+1} = Bx_k + c$, if $||B||_\infty < 1$, then $\lim_{k\to\infty} x_k = x$ for any initial guess x_0.

   (k) In solving a 2-point boundary value problem using a finite-difference scheme with mesh size h, if Jacobi's method is used to solve the linear system, then the spectral radius of the iteration matrix satisfies $\lim_{h\to 0} \rho(B_J) = 0$.

   (l) Suppose a two-dimensional boundary value problem is solved using a finite-difference scheme and the resulting linear system is solved by Jacobi's method with stopping criterion $||r_k||_\infty \leq 10^{-2}$. If the mesh size h is decreased, then the number of iterations needed to satisfy the stopping criterion is increased.

   (m) The method called "shifted inverse iteration" is used to find the inverse of a matrix.

   (n) Wilkinson's example shows that the coefficients of a polynomial can depend sensitively on the roots.

(o) When the power method is applied to find the largest eigenvalue and the corresponding eigenvector of a matrix, the vectors are normalized at each step in order to increase the rate of convergence of the method.

(p) If $p_n(x)$ is the interpolating polynomial of degree n for a given function $f(x)$ at points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$, then $\lim_{n\to\infty} p_n(x) = f(x)$ for all $x \in [a, b]$.

(q) If $p_n(x)$ is the Taylor polynomial of degree n for $f(x)$ about $x = a$, then $p_n^{(n+1)}(a) = 0$.

(r) If x_0, x_1, x_2 are three distinct points and $p(x) = L_0(x) + L_1(x) + L_2(x)$, then $p'(x_i) = 0$ for $i = 1, 2, 3$.

(s) The problem with polynomial interpolation at uniform points can be overcome by using double precision arithmetic.

(t) Chebyshev points are advantageous for polynomial interpolation because they are clustered near the center of the interval.

(u) Suppose $f(x)$ is approximated by a cubic spline interpolant $s(x)$ on the interval $a \le x \le b$ with interpolation points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$. Then if n is doubled, the error defined by $\max_{a\le x\le b} |f(x) - s(x)|$ is reduced by a factor of approximately 1/16.

2. State one advantage of ...

(a) Newton's method over the secant method.

(b) Gaussian elimination with pivoting over Gaussian elimination without pivoting.

(c) optimal SOR over Gauss-Seidel in solving.

(d) Rayleigh quotient iteration over the power method.

(e) polynomial interpolation over Taylor approximation.

(f) Newton's form for the interpolating polynomial over Lagrange's form.

(g) cubic spline interpolation over piecewise linear interpolation.

(h) cubic Hermite interpolation over cubic spline interpolation

3. Let $f(x) = e^x$ and let $D(h)$ denote the forward difference approximation of the first derivative, $f'(x) \approx D_+f(x)$ with $h = 1, \frac{1}{2}, \frac{1}{4}$. Apply Richardson extrapolation to obtain more accurate values.

4. Consider solving the equation $f(x) = x^2 - 5 = 0$ by the bisection method.

(a) Explain why $0 \le x \le 4$ is a suitable starting interval for this method.

(b) Take 3 steps of the bisection method, i.e. compute x_0, x_1, x_2.

(c) Approximately how many steps are needed to ensure that the error is less than 10^{-4}?

5. Suppose fixed-point iteration is applied to the function $g(x) = x^2 - \frac{1}{2}x + \frac{1}{2}$. Find the fixed points and in each case determine whether the iteration converges for starting values sufficiently close to the fixed point.

6. Which of the following matrices are positive definite?

   (a) $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$,

   (b) $\begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}$

7. Let $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$.

   (a) Find a vector x such that $||Ax||_\infty = ||A||_\infty$.

   (b) Find an approximate e-value of A by taking one step of the power method with initial guess $v^{(0)} = \frac{1}{\sqrt{3}}(1,1,1)^T$.

   (c) If $Ax = b$ is solved by Jacobi's method, does the iteration converge for any initial guess?

   (d) Answer the same question for Gauss-Seidel.

   (e) Explain how to find the optimal SOR parameter ω_*.

8. Prove the following results.

   (a) If A is positive definite, then A is invertible.

   (b) If A is positive definite, then the diagonal elements are positive.

   (c) If A is positive definite, then the eigenvalues are positive.

   (d) If A is invertible, then A^TA is symmetric and positive definite.

9. The SOR method in component form for a 2×2 system is given below. Find and correct any errors.

$$\begin{aligned} a_{11}x_1^{(k+1)} &= a_{11}x_1^{(k)} - \omega(a_{11}x_1^{(k)} + a_{12}x_2^{(k)} - b_1) \\ a_{22}x_2^{(k+1)} &= a_{22}x_2^{(k)} - \omega(a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} - b_2) \end{aligned}$$

10. The two-point boundary value problem $y'' - y = x, y(0) = 1, y(1) = 0$ for $0 \le x \le 1$ is solved by the finite-difference scheme $D_+D_-w_i - w_i = ih$ for $i = 1 : n$ with step size $h = 1/(n+1)$ and $w_0 = 1, w_{n+1} = 0$. Using $n = 3$, write down the linear system $A_h w_h = f_h$.

11. Let $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$. How many steps of Jacobi's method are required to reduce the error as much as one step of the Gauss-Seidel method?

12. Apply the spectral method to solve $Ax = b$, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

13. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find $\max\limits_{x \neq 0} R_A(x)$.

14. The following code is one of the eigenvalue algorithms we studied in class. What is the name of this algorithm? If it is applied to a 3×3 matrix A with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$, which eigenvalue does the method converge to?

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v = [1 1 1]'/sqrt(3);
for k = 1:kmax
lambda = v'*A*v;
w = A\v;
v = w/norm(w);
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15. Let $p(x) = 1 + x + x^2$ and take $x_0 = -1, x_1 = 0, x_2 = 1$. Write $p(x)$ in (a) Lagrange form, (b) Newton form, (c) nested form.

16. The outdoor temperature $T(t)$ is recorded at two-hour intervals starting at 8am and ending at 4pm, but the 12pm measurement was accidentally omitted. The recorded temperatures are $T(\text{8am}) = 30°\,\text{F}$, $T(\text{10am}) = 40°\,\text{F}$, $T(\text{2pm}) = 50°\,\text{F}$, $T(\text{4pm}) = 60°\,\text{F}$. Use a cubic interpolating polynomial to estimate the missing temperature.

17. Let $f(x)$ be a given function and let $T_1(x)$ denote the Taylor polynomial of degree 1 for $f(x)$ about a given point $x = a$. Show that $f(x) = T_1(x) + \int_a^x (x-t) f''(t)dt$. (This gives the integral form for the error in linear Taylor approximation.)

18. Find the natural cubic spline $s(x)$ satisfying $s(0) = 0, s(1/2) = 1, s(1) = 0$. Your answer will be 2 cubic polynomials, $s_0(x)$ defined on the interval $0 \leq x \leq \frac{1}{2}$ and $s_1(x)$ defined on the interval $\frac{1}{2} \leq x \leq 1$.

19. Consider the integral $I = \int_0^1 xe^{-x^2} dx$.
a) Compute I using the trapezoid rule and Richardson extrapolation with $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.
b) Estimate how small h must be for the trapezoid rule to give the same error as in the last column of the table.

20. The local form of the <u>midpoint rule</u> is $\int_0^h f(x)dx \sim cf(\frac{1}{2}h)$.
a) Find the value of the constant c which ensures that the midpoint rule is exact for constant functions. Show that the resulting method is also exact for linear functions.
b) Is the midpoint rule more accurate or less accurate than the trapezoid rule?

21. The Legendre polynomials $P_i(x), i = 0 : 3$ were derived in class.
a) Apply the Gram-Schmidt process to find $P_4(x)$.
b) Express x^4 as a linear combination of $P_i(x), i = 0 : 4$.

22. Evaluate the integral $\int_0^{2\pi} \mathrm{e}^{-x} \sin x \mathrm{d}x$ using the following methods.

    (a) trapezoid rule with $h = 2\pi, \pi, \frac{1}{2}\pi$

(b) Richardson extrapolation applied to the results in (a)

(c) three-point Gaussian quadrature

(d) the methods of Calculus I (e.g. integration by parts, FTC, etc.)

23. Among the functions $\{1, x, x^2, \sin \pi x, \cos \pi x, \sin^2 \pi x\}$, find all pairs that are orthogonal with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$.

24. Consider the function $f(t, y) = \frac{2}{t}y + t^2 e^t$.

(a) Show that $f(t, y)$ satisfies a Lipschitz condition on $D = \{(t, y) : 1 \leq t \leq 2, -2 \leq y \leq 5\}$.

(b) Show that $f(t, y)$ does not satisfy a Lipschitz condition for any interval containing $t = 0$.