Math 471-001 Fall 2013 Homework 6 due : Tuesday, Dec. 10

interpolation

1. page 351, problem 9

   Note: This problem asks you to derive an error bound for linear polynomial interpolation,

   $$|f(x) - p_1(x)| \leq \frac{1}{8} \max_{x_0 \leq x \leq x_1} |f''(x)| \, h^2, \text{ where } h = x_1 - x_0.$$

   You may prove this result by applying the theorem on the error in polynomial interpolation which was stated in class. The theorem says that given a function $f(x)$ and $n+1$ distinct points $a = x_0 < x_1 < \cdots < b = x_n$, then $f(x) = p_n(x) + \frac{1}{(n+1)!} f^{(n+1)}(\zeta)(x - x_0) \cdots (x - x_n)$, where $p_n(x)$ is a polynomial of degree n that interpolates $f(x)$ at the points x_i and ζ is some point in the interval $[a, b]$.

2. Write a Matlab program to perform natural cubic spline interpolation at the uniform points on the interval $-1 \leq x \leq 1$ for $f(x) = |x|$. In Matlab use $f(x) =$ `abs(x)`. You may use the backslash command to solve the linear system for the spline coefficients. Let $s_n(x)$ denote the spline based on n intervals and investigate the convergence of $s_n(x)$ to $f(x)$ by running the program for different values of n. In your writeup, include your code and plots of $s_n(x)$ and the function $f(x)$ for $n = 2, 4, 6$. Answer the following questions:

   (a) Does $s_n(x)$ converge pointwise to $f(x)$ on [-1,1]?

   (b) Does $s_n(x)$ converge uniformly to $f(x)$ on [-1,1]?

   notes:

   pointwise convergence means that $\lim_{n\to\infty} s_n(x) = f(x)$ for all $x \in [-1, 1]$

   uniform convergence means that $\lim_{n\to\infty} \max_{-1\leq x\leq 1} |s_n(x) - f(x)| = 0$

   Uniform convergence implies convergence, but the converse is false.

numerical integration

3. Let $f(x) = e^{-|x|}$ and consider the three points $x_0 = -1, x_1 = 0, x_2 = 1$.

   (a) Find Newton's form of the interpolating polynomial $p_2(x)$.

   (b) Plot $f(x)$ and $p_2(x)$ on the same graph and label each curve.

   (c) Compute $\int_{-1}^{1} f(x)\, dx$ and $\int_{-1}^{1} p_2(x)\, dx$

   (d) Approximate $\int_{-1}^{1} f(x)\, dx$ using three-point Gaussian quadrature

   (e) Compare the error in the integral of p_2 and Gaussian quadrature to the exact value of $\int_{-1}^{1} f(x)\, dx$.

initial value problems

4. Prove that for all $x \geq -1$ and $m > 0$,

$$0 \leq (1+x)^m \leq e^{mx}.$$

Note that this result was used to prove the convergence of one-step methods (like Euler's method) to the exact solution of linear and nonlinear first-order ODEs.

5. Consider the ODE $y'(t) = 1 - t + 4y$ with initial condition $y(0) = 1$.

(a) Show that the solution is $y(t) = \frac{1}{4}t - \frac{3}{16} + \frac{19}{16}e^{4t}$.

(b) Use Matlab to approximate this solution for $0 \leq t \leq 2$ using Euler's method with $\Delta t = 0.05, 0.025, 0.01$, and 0.001. Plot your results and the exact solution on the same set of axes in the t-y plane. Fill in the following tables

solutions					
t	approx $\Delta t = 0.05$	approx $\Delta t = 0.025$	approx $\Delta t = 0.01$	approx $\Delta t = 0.001$	exact $y(t)$
0	1	1	1	1	1
0.1					
0.2					
0.3					
0.4					
0.5					
1.0					
1.5					
2.0					

error				
t	$\Delta t = 0.05$	$\Delta t = 0.025$	$\Delta t = 0.01$	$\Delta t = 0.001$
0	0	0	0	0
0.5				
1.0				
1.5				
2.0				

6. The previous problem had an exact solution, which you hopefully used to make sure that your Euler's method program is working properly. Now apply the same program to an ODE whose exact solution cannot be represented analytically:

$$y'(t) = (t^2 - y^2)\sin y, \qquad y(0) = -1$$

Plot your solutions for $0 \leq t \leq 1$ using $\Delta t = 0.05$, 0.025, and fill in the table.

solutions		
t	$\Delta t = 0.05$	$\Delta t = 0.025$
0	-1	-1
0.1		
0.2		
0.3		
0.4		