PROBLEM SET 9 (DUE ON THURSDAY, NOV 10)

(All Exercises are references to the August 29, 2022 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Suppose that X is a closed subscheme of $\operatorname{Proj} S_\bullet$ for some graded ring $S_\bullet$ that is finitely generated by elements of degree 1 (as an S_0-algebra). Show that X is isomorphic to $\operatorname{Proj}(S_\bullet/I)$ for some homogeneous ideal I. (Hint: X will be cut out by a collection of ideals, each in a ring corresponding to one affine open $D(f)$, compatible with respect to restrictions to $D(fg) \subset D(f)$; you need to piece these ideals together to construct a single homogeneous ideal in $S_\bullet$.)

Problem 2. A *quadric* in $\mathbb{P}^n_k$ is a closed subscheme cut out by a single homogeneous polynomial of degree two (see 9.3.2). Give an example of two quadrics in $\mathbb{P}^2_{\mathbb{R}}$ intersecting in a single point, and compute the scheme-theoretic intersection. Then give a second example of this, with scheme-theoretic intersection not isomorphic (as schemes) to that in your first example. Then give a third example with intersection not isomorphic to either of the first two! (Changes from last week's problem: $\mathbb{P}^2$ instead of $\mathbb{A}^2$; $\mathbb{R}$ instead of $\mathbb{C}$; three examples instead of two examples. It may be helpful to note that Bezout's theorem now applies and says that the intersection must be of the form $\operatorname{Spec} A$, where A is an $\mathbb{R}$-algebra that is 4-dimensional as an $\mathbb{R}$-vector space and has exactly one prime ideal.)

Problem 3. Let $X = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0x_3 - x_1x_2)$, a quadric surface in $\mathbb{P}^3_{\mathbb{C}}$. Describe a closed embedding $\iota : \mathbb{P}^1_{\mathbb{C}} \to X$ with the property that the image of ι is not contained in any plane in $\mathbb{P}^3_{\mathbb{C}}$. Then describe a morphism $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ with the property that $\pi \circ \iota$ is the identity morphism on $\mathbb{P}^1_{\mathbb{C}}$. What do the fibers of π over closed points look like, as closed subsets of $\mathbb{P}^3_{\mathbb{C}}$? (Hint: Veronese embeddings and Exercise 9.3.L (rulings on the quadric surface) might both be helpful here.)