PROBLEM SET 11 (DUE ON THURSDAY, NOV 30)

(All Exercises are references to the July 31, 2023 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Suppose that X is an A-scheme. Show that X is separated as an A-scheme if and only if X is separated as a $\mathbb{Z}$-scheme.

Problem 2. Suppose $\sigma : X \to Y$ is a section of some morphism $\tau : Y \to X$, i.e. $\tau \circ \sigma$ is the identity on X. First show that if τ is separated, then σ is a closed embedding. Then modify your argument to show that even if τ is not separated, σ is a locally closed embedding. (You may want to read section 9.2 on locally closed embeddings, also you may want to use Proposition 11.3.1(b).)

Problem 3. Exercise 11.4.B (when are morphisms determined by where they send closed points? - you may want to read the preceding exercise/minor remarks and also look at Exercises 3.6.J (on an older homework) and 5.3.F (related))

Problem 4. Check that every rational map $\mathbb{A}^1_{\mathbb{C}} \dashrightarrow \mathbb{A}^1_{\mathbb{C}}$ can be extended to a morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. Then prove that (in contrast) the rational map $x/y : \mathbb{A}^2_{\mathbb{C}} \dashrightarrow \mathbb{A}^1_{\mathbb{C}}$ cannot be extended to a morphism $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. (One possible approach (though not the shortest): compute the graph of this rational map composed with the inclusion $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, as defined in 11.4.4, and observe that the result cannot be the graph of any morphism.)

Problem 5. Let $n \geq 2$ be an integer. Compute the (maximal) domain of definition of the generalized Cremona transformation

$$C : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}},$$

a rational map given by $[x_0 : \cdots : x_n] \mapsto [x_0^{-1} : \cdots : x_n^{-1}]$ (on closed points with $x_0 \cdots x_n \neq 0$ - your first task is to figure out how to construct such a map!).