PROBLEM SET 11 (DUE ON THURSDAY, NOV 21)

(All Exercises are references to the July 27, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Recall that any scheme has a unique morphism to $\operatorname{Spec}\mathbb{Z}$, so can naturally be viewed as a $\mathbb{Z}$-scheme. Suppose that X is an A-scheme. Show that X is separated as an A-scheme if and only if X is separated as a $\mathbb{Z}$-scheme.

Problem 2. Suppose $\sigma : X \to Y$ is a section of some morphism $\tau : Y \to X$, i.e. $\tau \circ \sigma$ is the identity on X. First show that if τ is separated, then σ is a closed embedding. Then modify your argument to show that even if τ is not separated, σ is a locally closed embedding. (You may want to read section 9.2 on locally closed embeddings since I only briefly mentioned them in class, also you may want to use Proposition 11.2.1(b).)

Problem 3. Exercise 11.3.B (when are morphisms determined by where they send closed points? - you may want to read the preceding exercise/minor remarks and also use the result of Exercise 5.3.F (a consequence of Exercises 3.6.J, which was on an earlier homework).

Problem 4. Let $n \geq 2$ be an integer. Compute the (maximal) domain of definition of the generalized Cremona transformation

$$C : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}},$$

a rational map given by $[x_0 : \cdots : x_n] \mapsto [x_0^{-1} : \cdots : x_n^{-1}]$ (on closed points with $x_0 \cdots x_n \neq 0$ - your first task is to figure out how to construct such a map!). (Hint: in general showing that a rational map $X \dashrightarrow Y$ has domain of definition U has two parts: constructing a morphism $\pi : U \to Y$ representing the rational map and showing that for each point $p \in X \setminus U$ and each open neighborhood V of p containing U, there does not exist a morphism $V \to Y$ representing the rational map. Two possible approaches for this latter part: first, you can sometimes show that there is no continuous extension of π to V by just looking at multiple different subsets of U with closure containing the point p. Second, you can pick an affine open neighborhood $\operatorname{Spec} A$ of p as well as an affine open cover $\{\operatorname{Spec} B_i\}$ of Y, and then you can try to show that there is never any morphism $D_{\operatorname{Spec} A}(f) \to \operatorname{Spec} B_i$ agreeing with π, where $D_{\operatorname{Spec} A}(f)$ is any distinguished open containing p. This can then be translated into algebra.)