PROBLEM SET 12 (DUE ON THURSDAY, DEC 5)

(All Exercises are references to the July 27, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 12.3.G (closed subvarieties of $\mathbb{P}^n$ intersect when suggested by dimensions - you will want to use some combination of the ideas used in Exercises 12.3.F, 12.3.D, 12.2.F (all of which you can cite without proof if you'd like) as well as the notion of an affine cone from section 9.3.11.)

Problem 2. Exercise 12.2.N (most surfaces of degree $d > 3$ have no lines - Vakil gives a detailed outline of how to do this, and you might also find it helpful to read his proof of Bertini's Theorem (13.4.2), but here is an additional note: if you are unfamiliar with the Grassmannian $\mathbb{G}(1,3)$, you can replace it in this proof with a single affine chart $\mathbb{A}^4$, where the closed point $(x_1, x_2, x_3, x_4) \in \mathbb{A}^4$ corresponds to the line between $[1 : 0 : x_1 : x_2]$ and $[0 : 1 : x_3 : x_4]$ in $\mathbb{P}^3$. You will conclude that "most" degree d surfaces have no lines of this form, and then you can finish by noting that the set of lines in $\mathbb{P}^3$ can be covered by finitely many charts of this type.)

Problem 3. The *tangent cone* at a point p of a scheme X is defined as $\operatorname{Spec} \bigoplus_{i \geq 0} \mathfrak{m}_p^i / \mathfrak{m}_p^{i+1}$, where $\mathfrak{m}_p$ is the maximal ideal in the local ring $\mathcal{O}_{X,p}$ and the direct sum is given a ring structure in the natural way. Let $X = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2)$ (two transverse lines) and $Y = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2 - x^3)$ (a nodal cubic curve). Show that X and Y have isomorphic tangent cones at the origin. (This is one way of making sense of the statement that these two curve singularities are the "same type" - a simple node. If you like number theory, you can similarly check that the tangent cone of $\operatorname{Spec} \mathbb{Z}[5i]$ at the point $[(5, 5i)]$ is isomorphic to the tangent cone of $\operatorname{Spec} \mathbb{F}_5[x, y]/(xy)$ at the origin.)