PROBLEM SET 5 (DUE ON THURSDAY, OCT 10)

(All Exercises are references to the July 27, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 5.1.B (irreducible closed subsets of general schemes are closures of points)

Problem 2. Suppose $f \in k[x_1, \ldots, x_n]$ is a polynomial and let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/f$. Prove that if X is irreducible and non-reduced, then X is not reduced at any point.

Problem 3. Let k be a field. Let $X = \operatorname{Spec} \mathbb{Z}[x, y]/xy$. There is a natural map $X(k[\epsilon]/\epsilon^2) \to X(k)$, where $X(A)$ is the set of A-valued points of X (i.e. morphisms from $\operatorname{Spec} A$ to X) - figure out how to define it. Compute the fibers of this map - you should find that for any $p \in X(k)$, the points in the fiber above p look like a k-vector space. What is the rank of this vector space, as a function of p? (This is a preview of the notion of the *tangent space* to a scheme at a point.)

Problem 4. Exercise 8.1.D (fiber products of open embeddings - there is a discussion of fiber products in Section 1.2.6)

Problem 5. We say that two integral schemes X, Y are *birational* (to each other) if there are nonempty opens $U \subseteq X$ and $V \subseteq Y$ such that U and V are isomorphic. This implies that their function fields $K(X), K(Y)$ are isomorphic.

(a) Give an example of two integral schemes X, Y which have isomorphic function fields but are not birational.

(b) Give an example of two integral schemes X, Y which are birational, but neither one is isomorphic to an open subscheme of the other.