PROBLEM SET 7 (DUE ON THURSDAY, OCT 24)

(All Exercises are references to the July 27, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Let $\pi : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{R}}$ be the morphism induced by the inclusion of graded rings $\mathbb{R}[x_0, x_1, \ldots, x_n] \to \mathbb{C}[x_0, x_1, \ldots, x_n]$. For each classical point $p = [a_0 : \ldots : a_n] \in \mathbb{P}^n_{\mathbb{C}}$, describe its image $\pi(p)$ (by e.g. giving generators for the corresponding homogeneous prime ideal). When do we have $\pi(p) = \pi(q)$ for two different classical points $p, q \in \mathbb{P}^n_{\mathbb{C}}$?

Problem 2. Let $X = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0x_3 - x_1x_2)$ (with all variables of degree 1 - note that this is *not* the same as the space from last week's homework), which you can think of as contained in $\mathbb{P}^3_{\mathbb{C}}$ via the closed embedding discussed in class induced by the quotient map of graded rings.

(a) Describe a closed embedding $i : \mathbb{P}^1_{\mathbb{C}} \to X$ with the property that its image is contained in some plane in $\mathbb{P}^3_{\mathbb{C}}$. (Here a plane is any closed set of the form $V(L) \subseteq \mathbb{P}^3_{\mathbb{C}}$, where L is a homogeneous polynomial of degree 1. For the purposes of this problem, you can just think about containment in terms of classical points if you want.)

(b) Now describe another closed embedding $j : \mathbb{P}^1_{\mathbb{C}} \to X$ with the property that its image is *not* contained in any plane in $\mathbb{P}^3_{\mathbb{C}}$.

(c) Now describe a morphism $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ with the property that $\pi \circ j$ is the identity morphism on $\mathbb{P}^1_{\mathbb{C}}$ (where j is the morphism you described in the previous part). What do the fibers of π over closed points look like, as closed subsets of $\mathbb{P}^3_{\mathbb{C}}$? (Hint: The morphisms i and j can both be constructed as induced morphisms from morphisms of graded rings, but the morphism π cannot be constructed from a single morphism of graded rings in this way. Exercise 9.3.L (rulings on the quadric surface) might be helpful for understanding the geometry in this problem.)