PROBLEM SET 8 (DUE ON THURSDAY, OCT 31)

(All Exercises are references to the July 27, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Let $S = k[x, y, t, u]/(xu - yt)$ be the graded ring considered in Problem 2 of Problem Set 6 (so $x, y \in S_1$ and $t, u \in S_0$). Let $\pi : \operatorname{Proj} S \to \operatorname{Spec} S_0 = \operatorname{Spec} k[t, u]$ be the morphism considered in that problem. Compute the scheme-theoretic fiber of π over the origin $[(t, u)] \in \operatorname{Spec} S_0$.

Problem 2. Exercise 10.2.J (distinct morphisms remain distinct upon extending the base field)

Problem 3. Suppose $f : X \to Z$ and $g : Y \to Z$ are morphisms, and let $X \times_Z Y$ be the fiber product. Also let $X^{\text{set}} \times Y^{\text{set}}$ be the Cartesian product of the underlying sets of X and Y. Define a function $p : X \times_Z Y \to X^{\text{set}} \times Y^{\text{set}}$ as the product of the two projections from the fiber product. Suppose that $x \in X, y \in Y, z \in Z$ are elements with $z = f(x) = g(y)$, so there are induced maps of residue fields $k_z \to k_x, k_z \to k_y$. Construct a bijection

$$p^{-1}((x, y)) \to \operatorname{Spec}\left(k_x \underset{k_z}{\otimes} k_y\right).$$

(Hint: By stacking Cartesian squares appropriately and interpreting things as fibers of morphisms, you should be able to construct a bijection from the left side to $\operatorname{Spec} k_x \times_Z \operatorname{Spec} k_y$. This is the same as $\operatorname{Spec} k_x \times_{\operatorname{Spec} k_z} \operatorname{Spec} k_y$ because $\operatorname{Spec} k_z \to Z$ is a monomorphism (left-cancellative). Feel free to assume this monomorphism fact without proof, or you can check it on an affine neighborhood of z with a little algebra.)

Problem 4. Describe two morphisms $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ such that the fiber product $X = \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{A}^1_{\mathbb{C}}} \mathbb{A}^1_{\mathbb{C}}$ using these morphisms has exactly two irreducible components and such that the two irreducible components intersect in exactly two points.