PROBLEM SET 13 (POSTED ON TUESDAY, DEC 2)

(All Exercises are references to the September 8, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 12.4.D (useful criterion for irreducibility - you will want to use properness to conclude that X is a variety and that some irreducible component of X surjects onto Y, and then use Theorem 12.4.1 to show that this is the only irreducible component)

Problem 2. The *tangent cone* at a point p of a scheme X is defined as $\operatorname{Spec} \bigoplus_{i\geq 0} \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}$, where $\mathfrak{m}_p$ is the maximal ideal in the local ring $\mathcal{O}_{X,p}$ and the direct sum is given a ring structure in the natural way. Let $X = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2)$ (two transverse lines) and $Y = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2 - x^3)$ (a nodal cubic curve). Show that X and Y have isomorphic tangent cones at the origin. (This is one way of making sense of the statement that these two curve singularities are the "same type" - a simple node. If you like number theory, you can similarly check that the tangent cone of $\operatorname{Spec} \mathbb{Z}[5i]$ at the point $[(5, 5i)]$ is isomorphic to the tangent cone of $\operatorname{Spec} \mathbb{F}_5[x,y]/(xy)$ at the origin.)

Problem 3. Suppose that X and Y are closed subvarieties of $\mathbb{P}^n_k$ of pure dimension d and $n - d$ respectively. Suppose that p is an isolated point (i.e. a connected component) in the intersection of X and Y, and suppose that X is singular at p. Show that the scheme-theoretic intersection $X \cap Y$ is not reduced at p. (Hint: Exercise 13.1.C might be helpful here.)