PROBLEM SET 5 (POSTED ON THURSDAY, OCT 9)

(All Exercises are references to the September 8, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 4.5.H(a) (bijection between homogeneous prime ideals in $S[\frac{1}{f}]$ and prime ideals in $S[\frac{1}{f}]_0$)

Problem 2. Let $S = k[x, y]/(x^2y)$ be a graded ring, where x and y have degree 1. Let $X = \operatorname{Proj} S$. Is X affine? Is X reduced?

Problem 3. Let $S = k[x, y, t, u]/(xu - yt)$ be a graded ring, where x, y have degree 1 and t, u have degree 0. (So $S_0 = k[t, u]$.)

(a) For any graded ring T, there is a natural morphism $\operatorname{Proj} T \to \operatorname{Spec} T_0$ coming from $(T[\frac{1}{f}])_0$ being a T_0-algebra. Let π be this morphism in the case $T = S$, i.e. $\pi : \operatorname{Proj} S \to \operatorname{Spec} S_0$. Check that the pullback map by π on rings of global sections $S_0 = \mathcal{O}_{\operatorname{Spec} S_0}(\operatorname{Spec} S_0) \to \mathcal{O}_{\operatorname{Proj} S}(\operatorname{Proj} S)$ is an isomorphism. (This is often (but not always) true for a general graded ring T - for instance it is true in the case of projective space $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$, where both sides will be k.)

(b) Let $\pi : \operatorname{Proj} S \to \operatorname{Spec} S_0 \cong \mathbb{A}^2_k$ be as above and let $U \subset \operatorname{Spec} S_0$ be the complement of the origin. Prove that π restricts to an isomorphism $\pi^{-1}(U) \cong U$.