

PROBLEM SET 8 (POSTED ON THURSDAY, OCT 23)

(All Exercises are references to the September 8, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

- Problem 1.** Let $S = k[x, y, t, u]/(xu - yt)$ be the graded ring considered in Problem 3 of Problem Set 6 (so $x, y \in S_1$ and $t, u \in S_0$). Let $\pi : \text{Proj } S \rightarrow \text{Spec } S_0 = \text{Spec } k[t, u]$ be the morphism considered in that problem. Compute the scheme-theoretic fiber of π over the origin $[(t, u)] \in \text{Spec } S_0$.
- Problem 2.** Exercise 10.2.J (distinct morphisms remain distinct upon extending the base field)
- Problem 3.** Suppose $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are morphisms, and let $X \times_Z Y$ be the fiber product. Also let $X^{\text{set}} \times Y^{\text{set}}$ be the Cartesian product of the underlying sets of X and Y . Define a function $p : X \times_Z Y \rightarrow X^{\text{set}} \times Y^{\text{set}}$ as the product of the two projections from the fiber product. Suppose that $x \in X, y \in Y, z \in Z$ are elements with $z = f(x) = g(y)$, so there are induced maps of residue fields $k_z \rightarrow k_x, k_z \rightarrow k_y$. Construct a bijection

$$p^{-1}((x, y)) \rightarrow \text{Spec} \left(k_x \otimes_{k_z} k_y \right).$$

(Hint: The argument was sketched in class - by stacking Cartesian squares appropriately and interpreting things as fibers of morphisms, with some thought you should be able to construct a bijection from the left side to $\text{Spec } k_x \times_Z \text{Spec } k_y$. To see that this is the same as $\text{Spec } k_x \times_{\text{Spec } k_z} \text{Spec } k_y$, you will want to prove that $\text{Spec } k_z \rightarrow Z$ is a monomorphism (left-cancellative). This can be checked on an affine neighborhood of z with a little algebra.)

- Problem 4.** Describe two morphisms $\mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ such that the fiber product $X = \mathbb{A}_{\mathbb{C}}^1 \times_{\mathbb{A}_{\mathbb{C}}^1} \mathbb{A}_{\mathbb{C}}^1$ using these morphisms has exactly two irreducible components and such that the two irreducible components intersect in exactly two points.