PROBLEM SET 9 (POSTED ON THURSDAY, OCT 30)

(All Exercises are references to the September 8, 2024 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 9.1.I(d) (an example of scheme-theoretic intersection not distributing over scheme-theoretic union)

Problem 2. Suppose that X is a closed subscheme of $\operatorname{Proj} S$ for some graded ring S that is finitely generated by elements of degree 1 (as an S_0-algebra). Show that X is isomorphic to $\operatorname{Proj}(S/I)$ for some homogeneous ideal I. (Hint: X can be cut out by a collection of ideals, each in a ring corresponding to one affine open $D(f)$, compatible with respect to restrictions to $D(fg) \subset D(f)$; you need to piece these ideals together to construct a single homogeneous ideal in S.)

Problem 3. A *quadric* in $\mathbb{A}^n_k$ is a closed subscheme $V(f)$ cut out by a single polynomial of degree two. Give an example of two quadrics in $\mathbb{A}^2_{\mathbb{C}}$ intersecting in a single point (and nowhere else), and compute the scheme-theoretic intersection. Then give a second example of this, with scheme-theoretic intersection not isomorphic (as $\mathbb{C}$-schemes) to that in your first example. Then give a third example with intersection not isomorphic to either of the first two! (Hint: you should mostly just try a bunch of examples, but you might also find thinking about Bezout's theorem to be helpful - it says that if you extend your quadrics to $\mathbb{P}^2_{\mathbb{C}}$ then they must intersect in exactly 4 points there, counted appropriately with multiplicities.)