

## PROBLEM SET 5 (POSTED ON THURSDAY, OCT 8)

(All Exercises are references to the October 21, 2025 version of *Foundations of Algebraic Geometry* by R. Vakil.)

- Problem 1.** Exercise 5.1.B (irreducible closed subsets of general schemes are closures of points)
- Problem 2.** Suppose  $f \in k[x_1, \dots, x_n]$  is a polynomial and let  $X = \operatorname{Spec} k[x_1, \dots, x_n]/f$ . Prove that if  $X$  is irreducible and non-reduced, then  $X$  is not reduced at any point.
- Problem 3.** Exercise 5.1.E (quasicompact schemes have closed points)
- Problem 4.** Let  $k$  be a field. Let  $X = \operatorname{Spec} \mathbb{Z}[x, y]/xy$ . There is a natural map  $X(k[\epsilon]/\epsilon^2) \rightarrow X(k)$ , where  $X(A)$  is the set of  $A$ -valued points of  $X$  (i.e. morphisms from  $\operatorname{Spec} A$  to  $X$ ) - figure out how to define it. Compute the fibers of this map - you should find that for any  $p \in X(k)$ , the points in the fiber above  $p$  look like a  $k$ -vector space. What is the rank of this vector space, as a function of  $p$ ? (This is a preview of the notion of the *tangent space* to a scheme at a point.)
- Problem 5.** Let  $X$  be a  $k$ -scheme. Prove (as claimed in class) that the natural map  $X(k) \rightarrow X$  sending a morphism (of  $k$ -schemes)  $\operatorname{Spec} k \rightarrow X$  to its image point is injective and has image equal to the set of closed points  $p \in X$  with residue field  $k_p = k$ .
- Problem 6.** Exercise 8.1.D (fiber products of open embeddings - there is a discussion of fiber products in Section 1.2.6)