# Math 632: Algebraic Geometry II

main website: www-personal.umich.edu/~pixton/632/

- syllabus
- link to Vakil's book
- problem sets (every 1-2 weeks)
- upcoming schedule
- notes from past lectures
- link to Canvas page
  - Zoom schedule/links for class/office hours
    (this week's office hours: W 2:30-3:30, F 4-5)
  - announcements, e.g. Later today I'll post a
    survey for office hour scheduling.

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graded components of class:

- 6-8ish problem sets
- near end of term: you will choose some additional
  topic/direction to read/learn about and then you will
  <u>either</u> write a short expository paper <u>or</u> do an
  informal oral exam with me.

math:

assuming Ch. 1–12 in Vakil, so a solid foundation in the category of schemes and properties of objects/morphisms

- affine schemes Spec A
- proj. schemes Proj S.
- general schemes, covered by affine opens

constructing new schemes from old schemes: fiber product, open subscheme, closed subscheme.

- many properties, e.g.
  - varieties:
    - finite type/k
    - reduced
    - separated
  - dimension, regularity: analogues of being a dim d manifold.

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This term: more subtle tools for studying schemes.

Previously we mostly understood a scheme $X$ through "$\underbrace{\text{sections}}$ of $\underbrace{\mathcal{O}_X}$" $\rightsquigarrow$ recovers $A$ from $\operatorname{Spec} A$.

replace by sheaf cohomology

replace by other sheaves, especially line bundles but also the cotangent sheaf!

$$H^i(X, \mathcal{F})$$

$$H^0(X, \mathcal{F}) = \mathcal{F}(X)$$

Some places where we'll apply these tools:

- understanding irreducible closed subsets of codim 1.
  $\longleftrightarrow$ understanding line bundles.
- classifying curves via their line bundles.
- distinguishing schemes via their cohomology / Euler characteristic.
- understanding local structure of morphisms via the relative cotangent sheaf.

We'll roughly cover Part V of Vakil, with some omissions and some additional topics.

Starting point: suitable generalizations of the structure sheaf $\mathcal{O}_X$ on a scheme $X$.

Def 1: An $\mathcal{O}_X$-module $\mathcal{F}$ is a sheaf of abelian groups on $X$ s.t. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module (and compatibility with restrictions)

Nice feature: The category of $\mathcal{O}_X$-modules is an abelian category.
(e.g. $\oplus$, ker, coker exist, good notion of exact sequence, etc.)

Bad feature: too many of these, in particular $\mathcal{F}$ is not determined by $\mathcal{F}(X)$ even for $X$ affine.

Example: $X = \underset{\text{Spec } k[t]}{\overset{\shortparallel}{\mathbb{A}^1_k}} \ni 0 = [(t)]$

$$\mathcal{F}(U) := \begin{cases} 0 & \text{if } U \ni 0 \\ \mathcal{O}_X(U) & \text{if } U \not\ni 0 \end{cases}$$

Want a notion closer to $\mathcal{O}_X$.

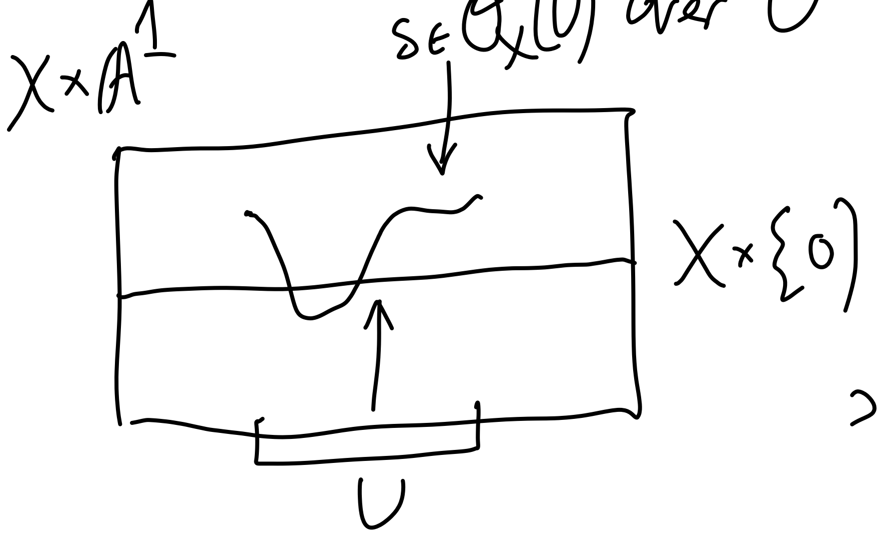
**Def 2**: A *line bundle* (or *invertible sheaf* or *locally free sheaf of rank 1*)

on $X$ is an $\mathcal{O}_X$-module $\mathcal{L}$ s.t. $X$ can be covered by opens $\{U_i\}$ with $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$.

Motivation for "line bundle" name:

if $X$ is a $k$-scheme, then

$$\mathcal{O}_X(U) = \left\{ (k\text{-scheme}) \text{ sections of the morphism } \begin{tikzcd} X \times \mathbb{A}^1 \arrow[d, "k"'] \\ X \arrow[u, dotted, bend right, "s"'] \end{tikzcd} \text{ over } U \right\}$$




so

expect a line bundle $\mathcal{L}$ to be the sheaf of sections of a morphism $Y \longrightarrow X$ that looks like $U \times \mathbb{A}^1 \to U$.

Nice features: closest to $\mathcal{O}_X$ in some sense

Less nice: not an abelian category:

1) not closed under $\oplus$, can "fix" this by

<u>Def 3</u>: A <u>vector bundle</u> (or <u>locally free sheaf</u>) of rank $n$ on $X$ is an $\mathcal{O}_X$-module $\mathcal{F}$ with $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n}$ for opens $U$ covering $X$.

2) not closed under cokernels.

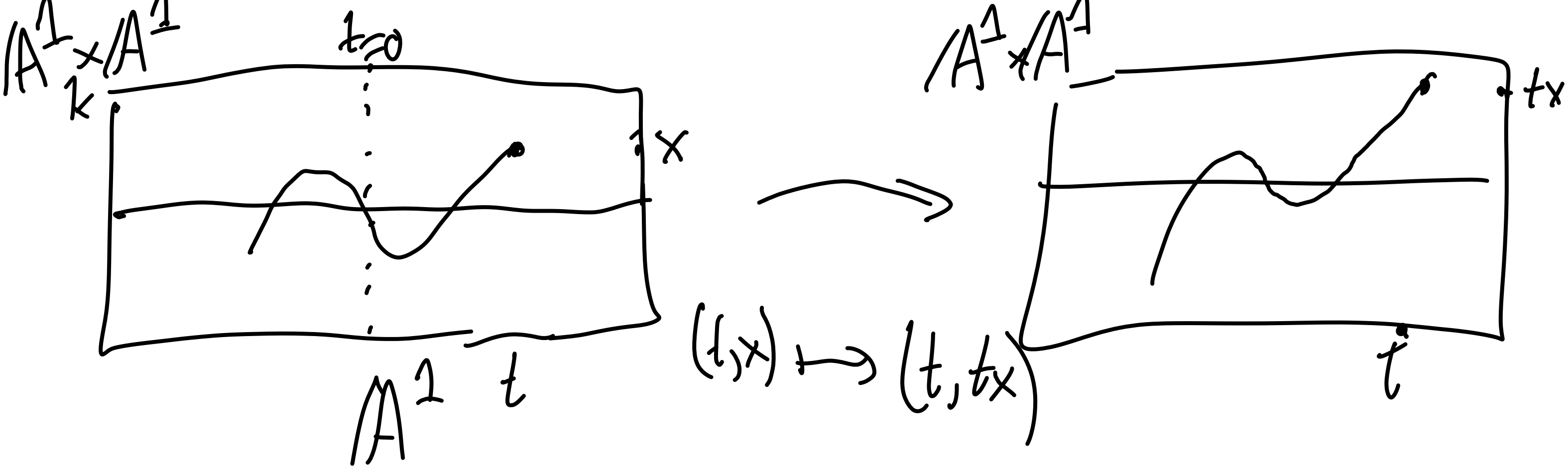
Example: $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$.

morphism $\alpha: \mathcal{O}_X \to \mathcal{O}_X$

$$\begin{array}{ccc} s & \longmapsto & ts \\ \cap & & \cap \\ \mathcal{O}_X(U) & & \mathcal{O}_X(U). \end{array}$$

$$\mathcal{F}(U) = \begin{cases} k & \text{if } U \ni 0 \\ 0 & \text{if } U \not\ni 0 \end{cases}$$

has cokernel isom to the skyscraper sheaf, which is nice, but not a vector bundle.




"maps on fibers" for this morphism $\mathcal{O}_X \to \mathcal{O}_X$:
usually rank 1, but rank 0 for $t=0$.

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4 defs of a quasicoherent sheaf on a scheme $X$:

Def 1) Quasicoherent sheaves are the smallest subclass of $\mathcal{O}_X$-modules that

vector bundle
a) contain $\mathcal{O}_X^{\oplus I}$
b) are local ($\mathcal{F}$ is in the class $\iff$ $\mathcal{F}|_{U_i}$ is for an open cover $U_i$)

c) form an abelian subcategory.

**Def 2:** An $\mathcal{O}_X$-module $\mathcal{F}$ is <u>quasicoherent</u> if

$$\mathcal{F}|_{U_i} \cong \operatorname{coker}\left(\alpha_i : \mathcal{O}_{U_i}^{\oplus I_i} \to \mathcal{O}_{U_i}^{\oplus J_i}\right)$$

for $U_i$ in an open cover and morphisms $\alpha_i$ between free sheaves $\mathcal{O}_{U_i}^{\oplus I}$.

"Locally a cokernel"

**Def 3:** An $\mathcal{O}_X$-module $\mathcal{F}$ is <u>quasicoherent</u> if for <u>any</u> affine open $\operatorname{Spec} A \subseteq X$,
(equiv. any affine open cover)

$$\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{M},$$ where

$\widetilde{M}$ is the $\mathcal{O}_{\operatorname{Spec} A}$-module constructed by $\widetilde{M}(D(f)) = M[\frac{1}{f}]$ where $M$ is some $A$-module.

**Cor:** The qcoh sheaves on $\operatorname{Spec} A$ correspond to $A$-modules.

Def 4: An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if

$$\mathcal{F}(\operatorname{Spec} A)\left[\tfrac{1}{f}\right] \cong \mathcal{F}\left(D_{\operatorname{Spec} A}(f)\right)$$

for all affine open $\operatorname{Spec} A \subseteq X$ and $f \in A$.