<u>So far</u>: understand qcoh sheaves via modules

<u>Today</u>: closed subschemes $\longleftrightarrow$ qcoh sheaves

$$Z \underset{\text{closed}}{\hookrightarrow} X \rightsquigarrow \text{ideal sheaf } \mathcal{I}_Z \subseteq \mathcal{O}_X$$

"functions vanishing on Z"

In the other direction, suppose $\mathcal{F}$ is a qcoh sheaf on X and suppose $s \in \mathcal{F}(X)$ is a global section.

Then we understand what

$$V(s) := \{x \in X \mid \underset{(\text{in } \mathcal{F}|_x)}{s(x) = 0}\}$$

is as a set.

Some issues with $V(s)$ in general:

(a) $V(s)$ isn't necessarily closed (e.g nonzero section of skyscraper sheaf)

(b) no natural scheme structure on $V(s)$

(c) depends on the choice of s

<u>Prop/Def</u>: If $\mathcal{F}$ is a rank r vector bundle on X and $s \in \mathcal{F}(X)$, then $V(s)$ can naturally be viewed as a closed subscheme of X.

<u>Pf</u>: On small open affines $\operatorname{Spec} A$ where $\mathcal{F}$ is trivialized, there exist isoms $\mathcal{F}|_{\operatorname{Spec} A} \cong \mathcal{O}_{\operatorname{Spec} A}^{\oplus r}$.

Under such an isom, $s|_{\operatorname{Spec} A} \in \mathcal{F}(\operatorname{Spec} A)$ corresponds to some $(f_1, \ldots, f_r) \in A^r$. Then can construct a closed subscheme $V(s)$ by gluing

$$\operatorname{Spec} A/(f_1, \ldots, f_r) \hookrightarrow \operatorname{Spec} A.$$

(Check: does not depend on isom, agrees as a set with earlier $V(s)$.) $\square$

$$\{\text{closed subschemes of } X\} \rightsquigarrow \{\text{qcoh sheaves on } X\}$$
$$Z \longmapsto \mathcal{I}_Z$$

$$\{\text{closed subscheme of } X\} \leftsquigarrow \left\{ (\mathcal{F}, s) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is a vector bundle on } X, \\ s \in \mathcal{F}(X) \end{array} \right\}$$

$$V(s) \longleftrightarrow (\mathcal{F}, s)$$

<u>Question</u>: When is $\mathcal{I}_Z$ a vector bundle?

<u>Lemma</u>: Let Z be a closed subscheme of X s.t. $X - Z$ is dense in X. If $\mathcal{I}_Z$ is a vector bundle, then $\mathcal{I}_Z$ is a line bundle and Z is locally cut out by a single function (i.e. $Z \cong V(f)$) that is not a zero divisor.

<u>Pf</u>: First, $\mathcal{I}_Z|_{(X-Z)} \cong \mathcal{O}_{(X-Z)}$, so $\mathcal{I}_Z$ is rank 1. Then on suff. small affine opens $\operatorname{Spec} A \subseteq X$, we have $\mathcal{I}_Z|_{\operatorname{Spec} A}$ corresponds to an ideal $I \subseteq A$ that is a free A-module of rank 1. ▨.

<u>Def</u>: A closed subscheme $Z \hookrightarrow X$ s.t. $\widetilde{I}_Z$ is a line bundle is called an <u>effective Cartier divisor</u>.

("divisor" in alg. geom. is shorthand for "codim 1")

(Here Z is codim 1 by Krull's principal ideal theorem since $Z \cong V(f)$ locally)

$$\left\{\text{effective Cartier divisors on } X\right\} \longrightarrow \left\{\text{line bundles on } X\right\}$$

$$D \longmapsto \widetilde{I}_D$$

$$\left\{\text{effective Cartier divisors on } X\right\} \longleftarrow \left\{(\mathcal{L}, s) \text{ on } X\right\}$$

$$V(s) \longleftarrow\!\shortmid (\mathcal{L}, s)$$

Would like to say these constructions are inverses, i.e.

$$\boxed{\widetilde{I}_{V(s)} \cong \mathcal{L}}\ ?$$ But this is false.

<u>Issue</u>: $\widetilde{I}_D$ usually don't have very many global sections

<u>Example</u>: If X is a connected reduced projective scheme over $k = \bar{k}$, then $\mathcal{O}_X(X) \cong k$ and $\mathcal{I}_D \subset \mathcal{O}_X$ means $\mathcal{I}_D(X) \subseteq \mathcal{O}_X(X) = k$, but constant functions don't vanish on D (for $D \neq \emptyset$) so actually get $\mathcal{I}_D(X) = 0$.

<u>Solution</u>: $\mathcal{I}_D$ doesn't necessarily have global sections, but $\mathcal{I}_D^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X)$ does: we have the global section corresp to the inclusion $\mathcal{I}_D \subseteq \mathcal{O}_X$.

Prop: Let X be an integral schee. Then there is a bijection

$$\left\{\begin{array}{c}\text{closed subschemes of } X \\ \text{loc. isom to } V(f) \text{ for } f\neq 0 \\ (\text{i.e. effective Cartier divisors})\end{array}\right\} \xleftrightarrow{\sim} \left\{(\mathcal{L}, s) \,\middle|\, \begin{array}{c}\mathcal{L} \text{ is a line bundle} \\ \text{on } X \text{ and} \\ s \in \mathcal{L}(X) - \{0\}\end{array}\right\} \Big/ \text{isom.}$$

$$D \longmapsto \left(\mathcal{I}_D^\vee,\ s = \mathcal{I}_D \hookrightarrow \mathcal{O}_X\right)$$

$$V(s) \longleftarrow (\mathcal{L}, s)$$

(i.e. need to check $\mathcal{I}_{V(s)}^\vee \cong \mathcal{L}$ and that $(\mathcal{I}_{V(s)} \hookrightarrow \mathcal{O}_X)$ corresp to s under this isom.)

This gives a different-looking geom. interpretation of the data of $(\mathcal{L}, s)$.

$$X = \mathbb{P}^1_k = \underset{\underset{\operatorname{Spec} k[\frac{y}{x}]}{\parallel}}{D(x)} \cup \underset{\underset{\operatorname{Spec} k[\frac{x}{y}]}{\parallel}}{D(y)}$$

$D = V(x^2) = \{[0:1]\}$ with some infinitesimal fuzz

$$= V(1) \cup V\left(\frac{x^2}{y^2}\right).$$

What is $\mathcal{I}_D^\vee$? Can compute on affines using modules,

i.e.

$$\mathcal{I}_D^\vee\big|_{\operatorname{Spec} k[\frac{x}{y}]} = \operatorname{Hom}_{k[\frac{x}{y}]}\left(k\left[\tfrac{x}{y}\right]\tfrac{x^2}{y^2}, k\left[\tfrac{x}{y}\right]\right)$$

$$= k\left[\tfrac{x}{y}\right]\text{"}\underbrace{\tfrac{y^2}{x^2}}\text{"} \quad \text{mult. by } \frac{y^2}{x^2}$$

and $\mathcal{I}_D^\vee\big|_{\operatorname{Spec} k[\frac{y}{x}]} = k\left[\frac{y}{x}\right] \cdot \text{"}1\text{"}$

Then the given global section is "1", formed by gluing

$\frac{x^2}{y^2} \cdot \text{"}\frac{y^2}{x^2}\text{"}$ to $1 \cdot \text{"}1\text{"}$.

Our proposition says that this section "1" of $\mathcal{I}_D^\vee$ should have vanishing scheme

$$V(\text{"}1\text{"}) = D = V(x^2).$$

The point is that $V(s)$ was defined by choosing trivializations, and on Spec $k[\frac{x}{y}]$ we had

$$\text{"}1\text{"} = \frac{x^2}{y^2} \cdot \text{"}\frac{y^2}{x^2}\text{"} \in k\left[\frac{x}{y}\right] \text{"}\frac{y^2}{x^2}\text{"},$$ so

it gets sent to $\frac{x^2}{y^2} \in k\left[\frac{x}{y}\right]$ under the isom.

($s=1$ may still vanish at points)

Another description of $\mathcal{I}_D^\vee$:

$$\mathcal{I}_D^\vee(U) = \left\{ \frac{f(x,y)}{x^2} \;\middle|\; \begin{array}{l} f \text{ is a homog. rat.} \\ \text{function of total degree } 2 \\ \text{with denom. not vanishing} \\ \text{on } U \end{array} \right\}$$

$$\cong \left\{ f(x,y) \mid f \text{ is } \ldots \right\} \cong \mathcal{O}_{\mathbb{P}^1_k}(2).$$

Two main flaws with this approach (effective Cartier divisors) to studying line bundles:

1) Some perfectly good line bundles (e.g. $\widetilde{I_D}$) have no nonzero global sections, so they don't contribute to the bijection. So we aren't seeing all of $\mathrm{Pic}(X)$, just some sub-semigroup.

2) Closed subschemes of X might be reducible and/or non-reduced, even if X itself is integral.

Plan: "Weil divisors" will be a solution to both issues (at least for nice X)

Idea: replace closed subschemes with

$\underbrace{\text{formal linear}}_{\substack{\text{includes negative coefficients,}\\ \text{not just positive}}}$ $\underbrace{\text{combinations}}_{\substack{\text{deal with}\\ \text{reducible}}}$ of irred closed $\underbrace{\text{subsets}}_{\text{of codim 1.}}$

not subschemes, deal with non-reduced.

(Replace the double point $V(x^2)$ by $2\{[0:1]\}$, etc)

__Def__: A __Weil divisor__ on X is a formal $\mathbb{Z}$-linear combination of irred. closed subsets of X of codim 1,

i.e. $$D = \sum_{Z \subset X} n_Z [Z] \text{ with } n_Z \in \mathbb{Z}, \text{ all but finitely many } n_Z = 0.$$

The set of Weil divisors, denoted $\operatorname{Weil} X$, is by construction a free abelian group.

group operation is $\otimes$.

<u>Goal:</u> (for nice X):

isomorphism $\mathrm{Weil}(X) \cong \{(\mathcal{L}, s) \mid \mathcal{L}$ line bundle, s is a nonzero rational section of $\mathcal{L}\}/\mathrm{iso}$

$\{\text{eff. Cartier divisors}\} \cong \{(\mathcal{L}, s) \mid s$ is a nonzero global section of $\mathcal{L}\}/\mathrm{iso}$

What is a rational section? Equivalence class of a pair $(U, s \in \mathcal{L}(U))$ where U is a dense open.