Last time:

$$\begin{tikzcd}
\left\{\text{effective Cartier divisors}\right\} \arrow[r, leftrightarrow, "\sim\ \checkmark"] \arrow[d, hook, "①"'] & \left\{(\mathcal{L}, s) \,\middle|\, \begin{array}{l} s \text{ a global section} \\ \text{nonzero} \end{array}\right\}/\text{iso} \arrow[d, hook, "\checkmark"] \\
\left\{\text{Weil divisors}\right\} \arrow[r, bend right, "③\ ``\mathcal{O}_X(D)"'] & \left\{(\mathcal{L}, s) \,\middle|\, \begin{array}{l} s \text{ a rational section} \\ \text{nonzero} \end{array}\right\}/\text{iso} \arrow[l, bend right, "②\ \mathrm{div}"'] \arrow[l, phantom, "\approx"]
\end{tikzcd}$$

$\{\text{Weil divisors}\} = \text{Weil } X$

$\underbrace{\left\{(\mathcal{L}, s) \mid s \text{ a rational section}\right\}/\text{iso}}_{\text{abelian group under } \otimes}$

Recall: a Weil divisor on X is

$$D = \sum_{\substack{Y \subset X \\ \text{codim} \\ \text{irred closed}}} n_Y [Y], \quad n_Y \in \mathbb{Z}, \text{ all but finitely many zero}$$

①: We'd like to define a map sending

$$\underset{\text{"double point"}}{V(x_0^2)} \hookrightarrow \mathbb{P}^1_k \quad \text{to} \quad 2[\{[0:1]\}] = 2[0:1]$$

So we want some notion of the multiplicity of a closed subscheme $Z \hookrightarrow X$ along a codim 1 closed irred set Y.

Assume that X is Noetherian and regular in codim 1. Let η_Y be the generic point of Y, so X is regular at η_Y.

Recall: X regular at codim 1 point η_Y
$\iff \mathcal{O}_{X,\eta_Y}$ '

Taking the stalk of the ideal sheaf $\mathcal{I}_Z$ at η_Y gives an ideal $\mathcal{I}_{Z,\eta_Y} \subseteq \mathcal{O}_{X,\eta_Y}$. Since $\mathcal{O}_{X,\eta_Y}$ is a DVR, we have $\mathcal{I}_{Z,\eta_Y} = m_{\eta_Y}^{a_Y}$ for some nonnegative integer a_Y.

This is the <u>multiplicity of Z along Y</u> and we get a Weil divisor $Z \rightsquigarrow \sum_Y a_Y[Y]$.

(can check: finite sum)

(2): similar to (1), but different input.

Again, assume X is Noetherian and regular in codim 1.

Instead of starting with general $(\mathcal{L}, s)$, let's consider $\mathcal{L} = \mathcal{O}_X$, so $s \in \mathcal{O}_X(U)$ for some dense open U.

For convenience, assume X is an integral scheme, so s, a rat. section of $\mathcal{O}_X$, can be viewed as an element of $K(X)^*$,

Then if Y is any $\underbrace{\text{codim 1 irred closed}}_{\text{Def: \underline{irred divisor}}}$ in X,

then $K(X) \cong K(\mathcal{O}_{X,\eta_Y})$ = field of fractions of a DVR,

so there's a valuation map

$$v_Y : K(X)^* \longrightarrow \mathbb{Z}.$$

Then can take $\sum_{Y \subset X} v_Y(s)[Y] \in \text{Weil } X.$

$$\underbrace{\text{If } s = \frac{f}{g} \in K(X) \text{ with } f, g \in \mathcal{O}_X(U), \text{ then "on } U\text{" we have that } v_Y(s) \neq 0 \Rightarrow Y \subset V(f) \cup V(g), \text{ so finitely many options by Noetherianity.}}_{\text{argument for } \sum_{Y \subset X} v_Y(s)[Y] \text{ being a finite sum.}}$$

This is the divisor (of zeroes and poles) of a rational section s of $\mathcal{O}_X$, denoted $\mathrm{div}(s)$.

Now suppose s is a rational section of a line bundle $\mathcal{L}$ on X. We want to define $v_Y(s) \in \mathbb{Z}$ for any irred divisor $Y \subset X$.

Choose an open nbhood U of η_Y and an isom $\mathcal{L}|_U \cong \mathcal{O}_U$. Then use prev def of $v_Y(s)$.

So we've defined a map

$$\left\{ (\mathcal{L}, s) \;\middle|\; \begin{array}{c} \mathcal{L} \text{ line bundle on } X \\ s \text{ a nonzero rat. section} \\ \text{of } \mathcal{L} \end{array} \right\} \xrightarrow{\text{div}} \text{Weil } X.$$

A little more work: this is a group homomorphism.

Need: $\text{div}(s_1 \otimes s_2) = \text{div}(s_1) + \text{div}(s_2)$,

where s_1 is a rat. section of $\mathcal{L}_1$,

s_2 is a rat. section of $\mathcal{L}_2$,

$s_1 \otimes s_2$ is a rat. section of $\mathcal{L}_1 \otimes \mathcal{L}_2$.

Check on trivialization; reduce to

DVR valuation is a group homomorphism.

Example:

$X = \mathbb{P}^1_k$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_k}(1)$, i.e.

$$\mathcal{L}(U) = \left\{ \text{homog. rat functions } \frac{f}{g} \text{ with } \deg f - \deg g = 1 \text{ and } g \neq 0 \text{ on } U \right\}.$$

Then $\dfrac{X_0^3}{(X_0 - X_1)(X_0 - 2X_1)}$ is a rat. section of $\mathcal{L}$,

with divisor $3[0:1] - [1:1] - [2:1]$.

Thm: Suppose X is a Noetherian normal scheme (recall: regular $\Rightarrow$ normal $\Rightarrow$ regular in codim 1).

Then div is injective.

So if s is a rat. section of a line bundle $\mathcal{L}$ on X as above, and if $\operatorname{div}(s) = 0$, then $(\mathcal{L}, s) \cong (\mathcal{O}_X, 1)$.

Pf: Can reduce to the case $X = \operatorname{Spec} A$ and $\mathcal{L} = \mathcal{O}_X$. Then it is "Algebraic Hartog's Lemma" (11.3.11 in Vakil):

Suppose A is an integrally closed Noetherian domain. Then

$$A = \bigcap_{\substack{\mathfrak{p} \subset A \\ \text{codim } 1}} A_{\mathfrak{p}} \quad (\text{inside } K(A)).$$

We would like some sort of inverse to div.

For the rest of today, we'll assume X is Noetherian and normal, and $\underbrace{\text{irreducible}}_{\text{for convenience only}}$.

Let $D \in \operatorname{Weil} X$. Then define a sheaf $\mathcal{O}_X(D)$ on X by

$$\underset{\substack{\text{alt. not.}\\ \Gamma(U,\, \mathcal{O}_X(D))}}{\left(\mathcal{O}_X(D)\right)(U)} := \left\{ f \in K(X)^* \;\middle|\; \underbrace{(\operatorname{div}(f) + D)\big|_U \geq 0}_{\text{condition on zeroes/poles of } f} \right\} \cup \{0\}.$$

where $|_U : \operatorname{Weil} X \to \operatorname{Weil} U$

$$[Y] \mapsto \begin{cases} [Y \cap U] & \text{if } Y \cap U \neq \emptyset \\ 0 & \text{else} \end{cases}$$

and $E \in \operatorname{Weil} X$ is "≥ 0" or "effective" if all of its coefficients are ≥ 0.

"f is allowed to have poles at positive parts of D and must have zeroes at negative parts of D."

<u>Example</u>: $X = \mathbb{P}^1_k$, $D = n[0:1]$, $(n>0)$

$$(\mathcal{O}_X(D))(U) = \left\{ \frac{f}{g} \text{ homog with } \deg f = \deg g \;\middle|\; \frac{g}{x_0^n} \text{ doesn't vanish on } U \right\}$$

$\mathcal{O}_X(D) \cong \mathcal{O}_{\mathbb{P}^1_k}(n)$ here.

What type of sheaf is $\mathcal{O}_X(D)$?

- easy to see that $\mathcal{O}_X(D)$ is a sheaf of abelian groups, and in fact an $\mathcal{O}_X$-module.
- somewhat trickier: $\mathcal{O}_X(D)$ is qcoh (want to compare sections on $\operatorname{Spec} A$ to $\operatorname{Spec} A[\frac{1}{f}]$)

But when is $\mathcal{O}_X(D)$ a line bundle?

(Note: $\mathcal{O}_X(D)$ has a canonical rational section called $\mathbb{1} \in K(X)^*$, defined on the open set U where $D|_U \geq 0$, i.e. throwing away the divisors where D has negative coefficient.

So if $\mathcal{O}_X(D)$ is a line bundle, would get a map

$$\begin{aligned} \mathrm{Weil}\, X &\longrightarrow \{(\mathcal{L}, s)\}/\text{iso} \\ D &\longmapsto (\mathcal{O}_X(D), \mathbb{1}). \end{aligned}$$

Prop: Let $\mathcal{L}$ be a line bundle and s a nonzero rat'l. section of $\mathcal{L}$. Then

$$\mathcal{O}_X(\mathrm{div}(s)) \cong \mathcal{L}$$ (and in particular is a line bundle).

Moreover,

$$(\mathcal{O}_X(\mathrm{div}(s)), 1) \cong (\mathcal{L}, s).$$

"$D \mapsto (\mathcal{O}_X(D), 1)$ is inverse to div"

Pf: next pset.

Remaining to do:

1) When is div an isomorphism?
2) descending to understand not just $\{(\mathcal{L}, s)\}/\mathrm{iso}$, but $\mathrm{Pic}(X) = \{\mathcal{L}\}/\mathrm{Isom}$.
3) actually computing $\mathrm{Pic}(X)$ in many cases.

Thursday.

$$\mathrm{Pic}(X) = \{\mathcal{L}\}/_{\text{isom}} \cong \frac{\{(\mathcal{L}, s)\}/_{\text{isom}}}{\{(\mathcal{O}_X, s)\}/_{\text{isom}}},$$

so if div is an isomorphism, we get

$$\mathrm{Pic}(X) \cong \mathrm{Weil}\, X / G \quad \text{for some}$$

subgroup G of Weil X.

Here G = group of divisors of elements of $K(X)^*$.