$$\begin{aligned} A^{n+1} &\longrightarrow \Gamma(X,\mathcal{L}) \\ e_i &\longmapsto s_i \end{aligned}$$

Last time:

$$\left\{ \begin{matrix} A\text{-scheme} \\ \text{morphisms } i: X \to \mathbb{P}^n_A \end{matrix} \right\} \longleftrightarrow \{(\mathcal{L}, \underbrace{s_0, \dots, s_n}_{\text{no common zeroes}})\}/\sim$$

Useful language (with $A = k$)

Def: A <u>linear system</u> (or <u>linear series</u>) on a k-scheme X is a k-vector space V along with a linear map $\lambda: V \longrightarrow \Gamma(X, \mathcal{L})$ for some line bundle $\mathcal{L}$.

A <u>base point</u> of a linear system is a point $p \in X$ where every section in the linear system vanishes.

The <u>base locus</u> is the set of base points. The linear system is base-point-free if the base locus is empty.

So finite rank base-point-free linear systems correspond to morphisms to $\mathbb{P}^n$.

Example: $X = \mathbb{A}^2_k$, $\mathcal{L} = \mathcal{O}_X$, V = linear system spanned by the sections x^2 and y^2 of $\mathcal{L}$.

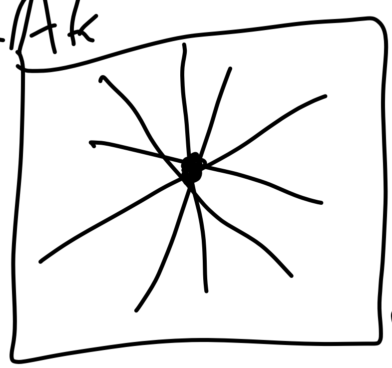
Then the base locus of V is $\{(0,0)\}$, so V corresponds to a morphism

$$\begin{aligned} \mathbb{A}^2_k - \{(0,0)\} &\longrightarrow \mathbb{P}^1_k \\ (x,y) &\longmapsto [x^2 : y^2] \end{aligned}$$

$\begin{array}{c} \text{linear system } V \\ \shortparallel \\ \langle x, y\rangle \end{array}$ $\longleftrightarrow$ collection of Weil divisors on X that are all linearly equiv.

$$\{\mathrm{div}(s) \mid \underset{s \neq 0}{s \in \mathrm{im}}(\lambda : V \to \Gamma(X, \mathcal{L}))\}$$

$X = \mathbb{A}^2_k$



divisors in linear system all contain base locus.

"base" = "stationary part of moving family".

__Def__: If $\lambda : V \to \Gamma(X, \mathcal{L})$ is an isom (of k-V.S.), we say that V is the __complete linear system__ of $\mathcal{L}$ and we use the notation $|\mathcal{L}|$ for V, or for the corresponding map

$$|\mathcal{L}| : X - (\text{base locus of } |\mathcal{L}|) \longrightarrow \mathbb{P}^n_k$$

if $\Gamma(X, \mathcal{L})$ is a finite rank k-vector space.

$$\left(n = \dim_k \Gamma(X, \mathcal{L}) - 1 \right)$$

(__Def__: The base locus of $\mathcal{L}$ is the base locus of $|\mathcal{L}|$).

(assume $\mathcal{L}$ base-point-free

$$|\mathcal{L}| : X \longrightarrow \mathbb{P}^n_k \qquad n = \dim_k \Gamma(X, \mathcal{L}) - 1$$

is defined by choosing a basis $s_0, \ldots, s_n$ for $\Gamma(X, \mathcal{L})$,

and changing which basis is used effectively composes $|\mathcal{L}|$ with an autom

$$\mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k \quad \text{(changing basis)}$$

Applications (of our bijection) to situations where we understand $\mathrm{Pic}(X)$:

1): $X = \mathbb{A}^m_k$, $\mathrm{Pic}(X) = 0 = \{\mathcal{O}_X\}$

Any morphism $\pi: \mathbb{A}^m_k \to \mathbb{P}^n_k$ is given by $n+1$ sections of $\mathcal{O}_X$ without a common zero, i.e.

$$(x_1, \ldots, x_m) \longmapsto [f_0 : \cdots : f_n] \text{ for}$$

polynomials $f_0, \ldots, f_n \in k[x_1, \ldots, x_m]$ without a common zero.

2): $X = \mathbb{P}^n_k$, $\mathrm{Pic}(X) = \{\mathcal{O}_{\mathbb{P}^n_k}(d) \mid d \in \mathbb{Z}\} \cong \mathbb{Z}$.

Suppose $\pi: \mathbb{P}^m_k \to \mathbb{P}^n_k$ corresponds to $\mathcal{L} = \mathcal{O}_X(d)$,

i.e. $\pi^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_{\mathbb{P}^m}(d)$ ("degree d morphism")

If $d < 0$, this is impossible because $|\mathcal{L}|$ is not base-point-free, since $\Gamma(X, \mathcal{L}) = 0$.

If $d = 0$, $\Gamma(X, \mathcal{L}) = k$, so a linear system is a functional $k^{n+1} \to k$

So we just get constant morphisms

$\pi: \mathbb{P}^m_k \to \mathbb{P}^n_k$, i.e. the image is a single closed k-valued point $[a_0 : \cdots : a_n]$.

If $d>0$, $\Gamma(X, \mathcal{L}) = \{$homog. polys in $X_0, \ldots, X_m$ of deg $d\}$, and π is of the form

$$[X_0 : \cdots : X_m] \longmapsto [f_0 : \cdots : f_n]$$ for homog. polys $f_0, \ldots, f_n$ of the same degree (and no common zero)

A nicer way to describe this: the complete linear system gives a morphism

$$|\mathcal{O}(d)| : \mathbb{P}^m_k \longrightarrow \mathbb{P}_k^{\binom{m+d}{d}-1}$$
$$[X_0 : \cdots : X_m] \longmapsto [X_0^d : X_0^{d-1}X_1 : \cdots : X_m^d]$$

$|\mathcal{O}(d)| = v_d$

Veronese embedding.

Any other morphism "of degree d"
$\pi : \mathbb{P}^m_k \to \mathbb{P}^n_k$ then factors as

$$\begin{tikzcd}
& \mathbb{P}^N_k \arrow[dr, dotted, "\exists!"] & \\
\mathbb{P}^m_k \arrow[ur, hook, "v_d"] \arrow[rr, "\pi"'] & & \underbrace{\mathbb{P}^n_k}_{\cong \mathbb{P}(V^\vee)}
\end{tikzcd}$$

rational map given by linear equations <u>dual</u> to the map $\lambda : V \to \Gamma(X, \mathcal{O}(d))$ defining the linear system.

<u>Moral</u>: the complete linear systems $|\mathcal{L}|$ are universal in the above sense.

If V ~~$\cong k^{n+1}$~~ is a f.d. k-vector space, we can define a graded ring $\mathrm{Sym}^* V = \bigoplus_{n \geq 0} \mathrm{Sym}^n V$.

If $V = ke_0 \oplus \cdots \oplus ke_n$, then $\mathrm{Sym}^* V = k[e_0, \ldots, e_n]$.

Then $\mathrm{Proj}(\mathrm{Sym}^* V) \cong \mathbb{P}^n_k$.

But $e_i \in V$ defines a hyperplane $\{e_i = 0\} \subset \mathbb{P}^n_k$, so $\mathrm{Proj}(\mathrm{Sym}^* V)$ is naturally viewed not as $\mathbb{P}V := (V - \{0\})/\text{scaling}$, but as $(V^\vee - \{0\})/\text{scaling}$.

3) $X = \mathbb{P}^{m_1}_k \times_{\operatorname{Spec} k} \mathbb{P}^{m_2}_k$ $\xrightarrow{pr_1} \mathbb{P}^{m_1}$, $\xrightarrow{pr_2} \mathbb{P}^{m_2}$

Problem set 2: $\operatorname{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$

$$\mathcal{O}(d_1, d_2) := pr_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(d_1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^{m_2}}(d_2) \in \operatorname{Pic}(X)$$

Can again understand all morphisms $X \to \mathbb{P}^n_k$ by thinking about the complete linear systems

$$|\mathcal{O}(d_1, d_2)|$$

Various cases:

$|\mathcal{O}(1,1)| : X \to \mathbb{P}^N$ ends up being the Segre embedding

$$\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \hookrightarrow \mathbb{P}^{m_1 m_2 + m_1 + m_2}.$$

You might enjoy thinking about

$|\mathcal{O}(1,0)|$ and $|\mathcal{O}(d_1, d_2)|$ for $d_1, d_2 > 0$.

4) $X = Bl_{(0,0)} \mathbb{A}^2_k$, $\operatorname{Pic}(X) \cong \mathbb{Z}$.

<u>Setup:</u> X is a k-scheme

$\mathcal{L}$ is a line bundle on X

$\Gamma(X, \mathcal{L})$ is finite rank (as a k-vector space)

If $\mathcal{L}$ is base-point-free, then we have a morphism $|\mathcal{L}|: X \to \mathbb{P}^N_k$. We've seen that sometimes $|\mathcal{L}|$ is a closed embedding (Veronese, Segre).

<u>Q:</u> When is $|\mathcal{L}|$ a closed embedding?

Equivalently, when is $(X, \mathcal{L}) \cong (\mathrm{Proj}\, S_\bullet, \mathcal{O}_{\mathrm{Proj}\, S_\bullet}(1))$ for some graded ring $S_\bullet$ f.g. in deg 1 over $S_0 = k$?

Def: Such a line bundle $\mathcal{L}$ on a proper k-scheme is known as **very ample**.

Examples: $\mathcal{O}_{\mathbb{P}^n}(d)$ is very ample $\iff d > 0$

$\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^n}(d_1, d_2)$ is very ample $\iff d_1, d_2 > 0$.

If $\mathcal{L}_1, \mathcal{L}_2$ are very ample, then so is $\mathcal{L}_1 \otimes \mathcal{L}_2$.

Intuition: being very ample is some sort of "positivity" property saying that there are lots of global sections — enough to distinguish any two points.

With this in mind, the Serre twist

$$\mathcal{F} \rightsquigarrow \mathcal{F} \otimes \mathcal{L}^{\otimes d} \quad \text{for } \underbrace{\mathcal{L} = \mathcal{O}_{\mathrm{Proj}\, S}(1)}_{\text{"}\mathcal{L} \text{ very ample"}}$$

should make $\mathcal{F}$ "more positive" for $d \gg 0$.

Def: A qcoh sheaf $\tilde{F}$ is globally generated if every stalk is generated by germs of global sections.

Examples: 1) Any qcoh sheaf on Spec A is globally generated.

2) A line bundle $\mathcal{L}$ is globally generated $\iff$ it is base-point-free.

Thm: Let $\mathcal{L}$ be a very ample line bundle on the proper k-scheme X.

$$\underbrace{}_{\text{(i.e. } X = \operatorname{Proj} S_\bullet,\ \mathcal{L} = \mathcal{O}_{\operatorname{Proj} S_\bullet}(1))}$$

Let $\mathcal{F}$ be a finite type qcoh sheaf on X.

Then for all suff. large d,

$\mathcal{F} \otimes \mathcal{L}^{\otimes d}$ is globally generated.

(stronger version in Vakil 16.6)