<u>Today</u>: some loose ends in qcoh sheaf thy (Ch. 17 in Vakil)

<u>Next week</u>: sheaf cohomology (Ch. 18)

<u>Some goals to keep in mind today</u>:

1) Given a vector bundle $\mathcal{F}$ on X, construct a geometric realization $\pi: E \to X$

2) We like proj schemes a lot, most recently because they come with line bundles $\mathcal{O}_{\text{Proj } S}(d)$. We'd now like a notion of projective morphism $\pi: X \to Y$ (should be a proj. A-scheme over each $\operatorname{Spec} A \underset{\text{open affine}}{\subseteq} Y$).

<u>Relative Spec</u>: Idea: a qcoh sheaf $\mathcal{F}$ on X might have the additional structure of not just being a sheaf of $\mathcal{O}_X$-modules, but a sheaf of $\mathcal{O}_X$-algebras.

Given such a qcoh sheaf of algebras $\mathcal{A}$ on X, we can apply Spec over each open affine and glue.

We get a scheme "relative Spec of $\mathcal{A}$"

$$\begin{tikzcd}
\underline{\mathrm{Spec}}\,\mathcal{A} \arrow[r] & X \\
\mathrm{Spec}\,\mathcal{A}(U) \arrow[u, phantom, "\cup"] \arrow[r] & U \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

U affine open

$\mathcal{A}(U)$ $\mathcal{O}_X(U)$-algebra

<u>Prop/Def</u>: This construction works.

<u>Examples:</u>

1) $\underline{\mathrm{Spec}}\,\mathcal{O}_X = id_X : X \to X$

2) $\underline{\mathrm{Spec}}(\mathcal{O}_X \times \mathcal{O}_X) = X \sqcup X$

3) Define $\mathcal{A}$ on X by

$$\mathcal{A}(U) := \mathcal{O}_X(U)[t]$$

Then $\underline{\mathrm{Spec}}\,\mathcal{A} = X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$

4) Suppose $\pi: Y \to X$ is a qcqs morphism. Then $\pi_* \mathcal{O}_Y$ is a qcoh sheaf of algebras on X, so we can consider $\underline{\mathrm{Spec}}\, \pi_* \mathcal{O}_Y$.

<u>Lemma</u>: If $\pi: Y \to X$ is an affine morphism, then $\underline{\mathrm{Spec}}\, \pi_* \mathcal{O}_Y \to X$ is isom to π.

$$\begin{tikzcd} \underline{\mathrm{Spec}}\, \pi_* \mathcal{O}_Y \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & X \\ Y \arrow[ur, "\pi"'] & \end{tikzcd}$$

(Easy to see that any $\underline{\mathrm{Spec}}\, \mathcal{A} \to X$ is an affine morphism)

A little more work: $\underline{\mathrm{Spec}}$ defines an equiv of categories

$$\left\{ \text{qcoh sheaves of } \mathcal{O}_X\text{-algebras} \right\} \underset{\pi_* \mathcal{O}_Y}{\overset{\underline{\mathrm{Spec}}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{affine morphisms} \\ Y \to X \end{array} \right\}$$

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[dr] & X \\ & Y_2 \arrow[u] \end{tikzcd}$$

morphisms $X \to \mathrm{Spec}\, A$

(*) $\longleftrightarrow$ morphisms $A \to \mathcal{O}_X(X)$

morphisms
$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\pi"'] & & \underline{\mathrm{Spec}}\, \mathcal{A} \arrow[dl] \\ & X & \end{tikzcd}$$

$\longleftrightarrow$ morphisms $\mathcal{A} \to \pi_* \mathcal{O}_Y$

5) Let $\mathcal{F}$ be a rank r vector bundle on a scheme X.

Consider the sheaf of algebras

$\mathcal{A} = \mathrm{Sym}^{*}(\mathcal{F}^{\vee})$ on X. Informally, for any affine open $U \subseteq X$ where $\mathcal{F}|_U \cong \mathcal{O}_X^{\oplus r}|_U$, take

$$\mathcal{A}(U) = \left\{ \begin{array}{l} \text{formal polynomials} \\ \text{with coeffs in} \\ \mathcal{O}_X(U) \end{array} \quad \begin{array}{c} \mathcal{F}(U) \longrightarrow \mathcal{O}_X(U) \\ \cong \\ \mathcal{O}_X(U)^{\oplus r} \end{array} \right\}$$

$$\cong \mathcal{O}_X(U)[t_1, \dots, t_r].$$

Then $p: \underline{\mathrm{Spec}}\,\mathcal{A} \longrightarrow X$ looks like $U \times \mathbb{A}^r$ over such U.

Moreover, a section of p over such a U (i.e. $s: U \to \underline{\mathrm{Spec}}\,\mathcal{A}$ with $p \circ s = \mathrm{id}_U$) corresponds to a ring homom (actually $\mathcal{O}_X(U)$-alg homog.)

$\mathcal{A}(U) \longrightarrow \mathcal{O}_X(U)$, which are naturally in correspondence with elements of $\mathcal{F}(U)$.

Claim: The sheaf of sections (as morphisms of schemes) of $p: \underline{\operatorname{Spec}}(\operatorname{Sym}^*(\mathcal{F}^\vee))$ is isomorphic to $\mathcal{F}$.

((Caution: only works for vector bundles, not for general qcoh $\mathcal{F}$.)

Relative Proj: Same idea: instead of looking like $\operatorname{Spec} B \to \operatorname{Spec} A$ locally, we want $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$ locally.

Prop/Def: Let $\mathcal{S}_\bullet$ be a qcoh sheaf of graded algebras on X. ($\mathcal{S}_\bullet = \bigoplus_{d\geq 0} \mathcal{S}_d$, $\mathcal{S}_0 = \mathcal{O}_X$).

Then we can construct a scheme

$$\begin{array}{ccc} \underline{\operatorname{Proj}}\, \mathcal{S}_\bullet & \longrightarrow & X \\ \cup\!\mid & & \cup\!\mid \text{ open affine} \\ \operatorname{Proj} \mathcal{S}(U)_\bullet & \longrightarrow & U \end{array}$$

Moreover, if $S_\bullet$ is <u>finitely generated in deg 1</u> (i.e. S_1 is finite type and $S(U)_\bullet$ is gen. in deg 1 for all affine open U), then the line bundles $\mathcal{O}_{\mathrm{Proj}\, S(U)_\bullet}(1)$ glue to define a line bundle $\mathcal{O}_{\underline{\mathrm{Proj}}\, S_\bullet}(1)$ on $\underline{\mathrm{Proj}}\, S_\bullet$.

Example/def: Let $\mathcal{F}$ be a finite type qcoh sheaf on X. Then its projectivization is

$$\mathbb{P}\mathcal{F} := \underline{\mathrm{Proj}}\,(\mathrm{Sym}^* \mathcal{F}).$$

When $\mathcal{F}$ is a rank r vector bundle, the map $\mathbb{P}\mathcal{F} \to X$ will locally look like

$$U \times \mathbb{P}^{r-1} \to U. \qquad \text{"projective bundle"}$$

Example: $X = \mathbb{P}^1$, $\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}(n)$ gives a sequence (for $n = 0, 1, 2, \ldots$) of interesting surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$, called the Hirzebruch surfaces.

Def: A morphism $\pi: Y \to X$ is projective if it is isomorphic to $\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$ for some qcoh sheaf of graded algs $\mathcal{S}$ that is f.g. in deg 1.

Warning: In general, this def fails all of the properties that we expect nice classes of morphisms to have:
(closed under composition, local on the target, closed under base chan.

In particular: a proj. morphism $Y \to \mathrm{Spec}\, A$ is the same thing as a projective A-scheme, but you need a little more to check whether a morphism is projective using an open cover.

(Idea of the obstruction: a projective A-scheme Y doesn't have a single canonical line bundle $\mathcal{O}(1)$ – depends on $\mathcal{S}$.
to glue, need to be able to pick compatible $\mathcal{O}(1)$'s on the cover.)

<u>Prop</u>: If X and Y are loc. Noetherian, $\pi: Y \to X$ is a morphism, and $\mathcal{L}$ is a line bundle on Y, then

"$\pi \cong \underline{\mathrm{Proj}}\, S^\bullet$ for some $S^\bullet$ with $\mathcal{L} \cong \mathcal{O}_{\underline{\mathrm{Proj}}\, S^\bullet}(1)$"

is an affine-local condition on X.

<u>(outlined in 17.3.7 in Vakil)</u>

If you like thinking of proj. A-schemes as closed subschemes of $\mathbb{P}^n_A$, you might like:

(Assuming $S^\bullet$ is f.g. in deg 1)

There is a closed embedding

$$\begin{tikzcd} \underline{\mathrm{Proj}}\, S^\bullet \arrow[r, hook] \arrow[dr] & \mathbb{P} S^1 \arrow[d] \\ & X \end{tikzcd} \qquad (\mathrm{Sym}^* S^1 \twoheadrightarrow S^\bullet)$$

If S^1 is a rank r vector bundle, then we have that the fibers of $\underline{\mathrm{Proj}}\, S^\bullet$ come with embeddings in $\mathbb{P}^{r-1}$.

Nice properties of projective morphisms.

1) Finite morphisms are projective.
(Pf: fully analogous to pf that finite A-schemes are projective A-schemes)

2) Projective morphisms are proper.
(since true over affine base and properness is local)

So: closed embedding $\Rightarrow$ finite $\Rightarrow$ projective $\Rightarrow$ proper.

3) The composition of proj. morphisms

$$X \xrightarrow[\text{proj}]{} Y \xrightarrow[\text{proj}]{} Z \text{ is proj's}$$

as Z is quasicompact.

<u>Cor</u>: If $X \to Y$ is a finite morphism and Y is a proj. A-scheme, then so is X.

(17.4 has applications to curves, delaying until a bit later along with other curve stuff.