<u>Today:</u> sheaf cohomology – properties, example application
<u>Thursday:</u> – construction, computations

---

tool with two main purposes:

1) compute spaces of global sections $\mathcal{F}(X)$
2) provide additional global invariants of vector bundles:
   locally vector bundles are trivial, so want global measures of their behavior, and $\mathcal{F}(X)$ is just one module.

Recall: If $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is a short exact sequence of sheaves of abelian groups,

we have

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\alpha(X)} \mathcal{G}(X) \xrightarrow{\beta(X)} \mathcal{H}(X), \text{ but}$$

$\beta(X)$ will not necessarily be surjective.

In other words, the functor $\Gamma(X, -): Ab_X \to Ab$ is left exact but not right exact.

Cohomology is a measure of this failure of exactness:

we will have an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \xrightarrow{\beta(X)} \mathcal{H}(X) \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to H^1(X,\mathcal{H}) \to H^2(X,\mathcal{F}) \to \cdots$$

"long exact sequence"

---

2 approaches to constructing/defining $\overbrace{H^i}^{\text{Čech}}\overbrace{(X,\mathcal{F})}^{\text{derived functors}}$ $(i \geq 0)$

1) derived functors: general machinery that can be applied to any left exact functor from a category that is "large enough"

We won't take this approach, but it is in Ch. 23 of Vakil.

2) Čech cohomology: construction in the special case of the functor of global sections on a sheaf

- takes on an especially simple form for qcoh sheaves on schemes (by affine open covers).

Basic properties: (standing assumptions: X is a separated and quasicompact A-scheme)

1) For each $i \geq 0$ there is an additive functor

$$H^i(X,-) : \underbrace{QCoh_X}_{\text{qcoh sheaves on } X} \longrightarrow \underbrace{Mod_A}_{A\text{-modules}}$$

with $H^0(X,-) = \Gamma(X,-)$.

2) Whenever $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is a s.e.s in $QCoh_X$, there is a long exact sequence

$$0 \longrightarrow H^0(X,\mathcal{F}) \xrightarrow{\alpha_*} H^0(X,\mathcal{G}) \xrightarrow{\beta_*} H^0(X,\mathcal{H}) \to$$

$$\to H^1(X,\mathcal{F}) \xrightarrow{\alpha_*} H^1(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{H}) \to$$

$$\to \cdots$$

3) If $\pi: X \to Y$ is a morphism (of A-schemes), then we know we have an isomorphism

$$H^0(Y, \pi_* \mathcal{F}) \to H^0(X, \mathcal{F}).$$

In general, there are natural maps

$$H^i(Y, \pi_* \mathcal{F}) \to H^i(X, \mathcal{F})$$

"natural" = commutes with functorial maps in $\mathcal{F}$.

<u>Properties specific to $QCoh_X$:</u>

4) If X is affine, then $H^i(X, \mathcal{F}) = 0$ for any $i > 0$ (and $\mathcal{F} \in QCoh_X$).

(Why? $\Gamma(X, -)$ is an exact functor here).

More generally, if X can be covered by $n+1$ affine opens, then $H^i(X, \mathcal{F}) = 0$ for $i > n$.

5) If $\pi: X \to Y$ is an affine morphism, then

$$H^i(Y, \pi_* \mathcal{F}) \to H^i(X, \mathcal{F}) \text{ is an isom}$$

for all i, not just $i = 0$.

Example of using cohomology:

Thm: Let A be a Noetherian ring. Let X be a projective A-scheme. Let $\mathcal{F}$ be a finite type qcoh sheaf on X. Then $\mathcal{F}(X)$ is a f.g. A-module.

(Note: not obvious even for $\mathcal{F} = \mathcal{O}_X$)

In fact, we'll prove more generally that

$$H^i(X, \mathcal{F}) \text{ is a f.g. } A\text{-module for all } i \geq 0.$$

($i=0$ is the hardest case!).

Pf of thm (really of):

Step 1: reduce to the case $X = \mathbb{P}^n_A$:

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n_A, j_* \mathcal{F}) \text{ for a}$$

$\underbrace{\text{closed embedding}}_{\text{affine morphism}}$ $j : X \hookrightarrow \mathbb{P}^n_A$

Step 2: check by explicit computation for

$$\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n_A}(d),\ d \in \mathbb{Z}.$$

We've done this when $i=0$, but $i>0$ will have to wait until we've constructed sheaf cohomology.

Step 3: find a s.e.s of the form

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(d)^{\oplus r} \longrightarrow \mathcal{F} \longrightarrow 0$$

for some $d \in \mathbb{Z}$, $r \geq 0$.

(discussion of this in 15.3 and 16.6 in Vakil)

This is implied by

Claim: $\mathcal{F}(d) \left(= \mathcal{F} \otimes \mathcal{O}(d)\right)$ is finitely globally generated for d suff. large (and $\mathcal{F}$ finite type on $X = \mathbb{P}^n_A$)

since this can be interpreted as saying that there is a surjection of sheaves

$$\mathcal{O}^{\oplus r} \longrightarrow \mathcal{F}(d) \longrightarrow 0,$$ and

then apply $\otimes\, \mathcal{O}(-d)$ to get.

$$\mathcal{O}(-d)^{\oplus r} \longrightarrow \mathcal{F} \longrightarrow 0.$$

<u>Pf of claim</u> (that $\mathcal{F}(d)$ is finitely glob. gen for $d \gg 0$):

Let $U_i = D(x_i)$ be the usual cover of $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots x_n]$.

Then $\mathcal{F}|_{U_i}$ is finitely globally generated since U_i is affine, say by sections $\underbrace{s_j^{(i)}}_{j} \in \Gamma(U_i, \mathcal{F}|_{U_i})$.

Then interpreting x_i as a section of $\mathcal{O}_{\mathbb{P}^n_A}(1)$, we have (by pset 1 problem) that each $s_j^{(i)}$ can be extended to a global section of some $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_A}(1)^{\otimes d_j^{(i)}}$ after mult by $x_i^{d_j^{(i)}}$. Take $d = \max(d_j^{(i)})$, and then $\{ s_j^{(i)} x_i^d \}$ generate the sle $\mathcal{F}(d)$. ∎.

<u>Step 4:</u> examine the long exact sequence corresponding to the s.e.s. in step 3, then first prove

$H^n(X, \mathcal{F})$ is f.g., then $H^{n-1}(X, \mathcal{F})$ is f.g., all the way down to $H^0(X, \mathcal{F})$ is f.g.

$$0 \to H^0(\mathbb{P}^n, \mathcal{G}) \to H^0(\mathbb{P}^n, \mathcal{O}(d)^{\oplus r}) \to H^0(\mathbb{P}^n, \mathcal{F}) \quad \boxed{\text{f.g.}}$$

$$\to \cdots$$

$$\to H^{n-1}(\mathbb{P}^n, \mathcal{G}) \to H^{n-1}(\mathbb{P}^n, \mathcal{O}(d)^{\oplus r}) \to H^{n-1}(\mathbb{P}^n, \mathcal{F}) \quad \text{③ f.g.}$$

$$\to H^n(\mathbb{P}^n, \mathcal{G}) \to H^n(\mathbb{P}^n, \mathcal{O}(d)^{\oplus r}) \to H^n(\mathbb{P}^n, \mathcal{F})$$

$$\to H^{n+1}(\mathbb{P}^n, \mathcal{G}) = 0$$

② f.g.

this column is f.g. by step 2

① f.g.

because $\mathbb{P}^n$ can be covered by $n+1$ open affines

This completes the proof assuming the existence of sheaf cohomology satisfying all the given properties and the computation in step 2 (that $H^i(\mathbb{P}^n_A, \mathcal{O}(d))$ is a f.g. A-module). $\square$