Last time:

Thm: A = Noetherian ring, X = proj. A-scheme, $\mathcal{F}$ = finite type qcoh sheaf on X.
Then $H^i(X, \mathcal{F})$ is a f.g. A-module for all $i \geq 0$.

Cor: Suppose $\pi: X \to Y$ is a proj. morphism of loc. Noetherian schemes. Then if $\mathcal{F}$ is a finite type qcoh sheaf on X, then
$\pi_* \mathcal{F}$ is a finite type qcoh sheaf on Y.

Cor: Suppose Y is loc. Noetherian. Then for a morphism $\pi: X \to Y$, finite $\iff$ affine + projective.

Today: fill in 2 gaps in pf of Thm: construction of $H^i(X, \mathcal{F})$ and f.g.-ness of $H^i(\mathbb{P}^n_A, \mathcal{O}(d))$.

$\check{\text{C}}$ech cohomology: (standing assumptions: X is a separated and quasicompact A-sch.)

Suppose $\mathcal{U} = \{U_j\} = \{U_0, \ldots, U_n\}$ is a finite cover of X by affine opens.

We will define functors $H^i_{\mathcal{U}}(X, -) : QCoh_X \to Mod_A$.

It will turn out later that the choice of $\mathcal{U}$ doesn't matter, so later $H^i := H^i_{\mathcal{U}}$ (for any $\mathcal{U}$).

Def: The $\check{\text{C}}$ech complex associated to a qcoh sheaf $\mathcal{F}$ and $\underset{\{U_0,\ldots,U_n\}}{\mathcal{U}}$ as above is:

$$0 \xrightarrow{\delta^0} \underbrace{\prod_{|I|=1} \mathcal{F}(U_I)}_{H^0} \xrightarrow{\delta^1} \underbrace{\prod_{|I|=2} \mathcal{F}(U_I)}_{H^1} \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^n} \underbrace{\prod_{|I|=n+1} \mathcal{F}(U_I)}_{H^n} \xrightarrow{\delta^{n+1}} 0 \;,$$

where $U_I = \bigcap_{j \in I} U_j$ for $I \subseteq \{0, \ldots, n\}$ and the map from $\mathcal{F}(U_I)$ to $\mathcal{F}(U_J)$ is $\begin{cases} (-1)^{k-1} \operatorname{res}_{U_I, U_J} & \text{if } J = I \cup \{j\},\ j \text{ is the } k\text{th element of } J. \\ 0 & \text{otherwise.} \end{cases}$

(complex of A-modules)

<u>Def</u>: $H^i_{\mathcal{U}}(X,\mathcal{F}) := \ker \delta^{i+1} / \operatorname{im} \delta^i$

(easy check: $\delta^{i+1} \circ \delta^i = 0$, so $\operatorname{im} \delta^i \subseteq \ker \delta^{i+1}$).

<u>Examples</u>:

1) $H^0_{\mathcal{U}}(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F})$ is just sheaf axioms.

(Aside: the augmented Čech complex starts

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_{|I|=1} \mathcal{F}(U_I) \longrightarrow \cdots$$

and this is exact)

2) $X = \mathbb{A}^2_k - \text{origin}$, $\mathcal{F} = \mathcal{O}_X$, $\mathcal{U} = \{D(x), D(y)\}$:

$$0 \longrightarrow k[x,x^{-1},y] \oplus k[x,y,y^{-1}] \xrightarrow{\delta_1} k[x,x^{-1},y,y^{-1}] \to 0$$

$$(f,g) \longmapsto g-f$$

$$H^1_{\mathcal{U}}(X,\mathcal{O}_X) = \operatorname{coker} \delta_1 = \bigoplus_{m,n\geq 1} k \cdot \frac{1}{x^m y^n} \neq 0.$$

Properties of $H^i_{\mathcal{U}}(X, \mathcal{F})$:

1) functorial in $\mathcal{F}$, since $\mathcal{F} \to \mathcal{G}$ induces a map of Čech complexes

$$\begin{tikzcd}
\cdots \arrow[r] & \prod\limits_{|I|=i} \mathcal{F}(U_I) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod\limits_{|I|=i+1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
& \prod\limits_{|I|=i} \mathcal{G}(U_I) \arrow[r] & \prod\limits_{|I|=i+1} \mathcal{G}(U_I) \arrow[r] & \cdots
\end{tikzcd}$$

and then a map $H^i_{\mathcal{U}}(X, \mathcal{F}) \to H^i_{\mathcal{U}}(X, \mathcal{G})$

2) Long exact sequence (apply Snake Lemma to short exact sequence of Čech complexes given)

3) $H^i_{\mathcal{U}}$ vanishes for $i \geq |\mathcal{U}|$.

4) preserved by pushforward by affine morphisms, i.e. if $\pi: X \to Y$ is affine and $\{U_j\}$ is an affine open cover of Y, then

$$H^i_{\{\pi^{-1}(U_j)\}}(X, \mathcal{F}) \cong H^i_{\{U_j\}}(Y, \pi_* \mathcal{F})$$

5) If $\{U_j\} \subseteq \{V_j\}$ are both affine open covers of X (with compatible orderings), then there is a map of Čech complexes

$$\prod_{|I|=i} \mathcal{F}(V_I) \longrightarrow \prod_{|I|=i} \mathcal{F}(U_i)$$

(proj. onto some components of product),

so an induced map $H^i_{\{V_j\}}(X,\mathcal{F}) \to H^i_{\{U_j\}}(X,\mathcal{F})$.

Thm: This map is an isomorphism (of A-modules).

Cor/Def: canonical identifications

$$H^i_{\{U_j\}} \xleftarrow{\sim} H^i_{\{U_j\}\cup\{U_j'\}} \xrightarrow{\sim} H^i_{\{U_j'\}}$$

can define $H^i = H^i_{\mathcal{U}}$ for any $\mathcal{U}$.

This Thm is fairly involved to prove. (18.2 in Vakil, will say a few words about it later).

Example: $X = \mathbb{P}^n_A = \operatorname{Proj} S_\bullet$, $S = A[x_0, \dots, x_n]$

affine open cover $U_i = D(x_i)$

$\mathcal{F} = \mathcal{O}(d)$. (for some $d \in \mathbb{Z}$)

The Čech complex is then the dth graded piece of the complex of graded $S_\bullet$-modules

$$0 \longrightarrow \prod_{|I|=1} S\left[\left\{\tfrac{1}{x_i} \,\middle|\, i \in I\right\}\right] \longrightarrow \prod_{|I|=2} S\left[\left\{\tfrac{1}{x_i} \,\middle|\, i \in I\right\}\right] \longrightarrow \dots$$

Claim: This complex is exact except at the two ends, where the "cohomology groups" (ker/im) are isomorphic to

$S_\bullet$ and $A[x_0^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ / sub-$S_\bullet$-module gen by monomials $x_0^{a_0} \cdots x_n^{a_n}$ with at least one $a_i \geq 0$.

Cor: $H^i(\mathbb{P}^n_A, \mathcal{O}(d)) \cong \begin{cases} A^{\binom{d+n}{n}} & \text{if } i=0 \text{ and } d \geq 0 \\ A^{\binom{-d-1}{n}} & \text{if } i=n \text{ and } d \leq -n-1 \\ 0 & \text{else.} \end{cases}$

Pf of Claim: induct on n and use a s.e.s. of complexes $(I \subseteq \{0, \dots, n\})$

$$\begin{tikzcd}
0 \arrow[r] & \prod_{\substack{|I|=1 \\ n\in I}} S[\frac{1}{x_I}] \arrow[r] \arrow[d] & \cdots \arrow[r] & \boxed{\prod_{\substack{|I|=n+1 \\ n\in I}} S[\frac{1}{x_I}]} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \boxed{\prod_{|I|=1} S[\frac{1}{x_I}]} \arrow[r] \arrow[d] & \boxed{\cancel{\cdots}} \quad \boxed{} \arrow[r] & \boxed{\prod_{|I|=n+1} S[\frac{1}{x_I}]} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \boxed{\prod_{\substack{|I|=1 \\ n\notin I}} S[\frac{1}{x_I}]} \arrow[r] & \cdots \quad \boxed{} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Snake Lemma maps in blue

red box = might have horizontal cohomology

complex 1 is the analogous Čech complex for $\mathbb{P}^{n-1}$ with $\otimes_A A[x_n^{\pm 1}]$ applied. and augmented with S at beginning

complex 3 is " " " " " " " " " " ... with $\otimes_A A[x_n]$ applied

The 4-term exact sequence on the right then is

$$0 \longrightarrow \bigoplus_{d\in\mathbb{Z}} \underbrace{H^{n-1}(\mathbb{P}^n, \mathcal{O}(d))}_{} \longrightarrow A[x_0^{-1}, x_1^{-1}, \dots, x_{n-1}^{-1}, x_n]$$

$$\longrightarrow A[x_0^{-1}, \dots, x_{n-1}^{-1}, x_n^{\pm 1}] \cdot x_0^{-1} \cdots x_n^{-1} \longrightarrow \bigoplus_{d\in\mathbb{Z}} H^n(\mathbb{P}^n, \mathcal{O}(d)) \xrightarrow{\cdot x_0^{-1} \cdots x_n^{-1}} 0.$$

with $H^{n-1}(\mathbb{P}^n, \mathcal{O}(d)) = 0$.

Key ingredient to the proof that

$H^i_{\mathcal{U}}(X, \mathcal{F})$ doesn't depend on $\mathcal{U}$ is that

$$H^i_{\mathcal{U}}(\operatorname{Spec} A, \mathcal{F}) = 0 \text{ for } i > 0$$

for any affine open cover $\mathcal{U}$ of $\operatorname{Spec} A$.