This week: X proj k-scheme, $\mathcal{F}$ finite type qcoh on X

$\rightsquigarrow$ $H^i(X,\mathcal{F})$ is a finite rank k-vector spaces.

Thm (last week)

Def: $h^i(X,\mathcal{F}) = \dim_k H^i(X,\mathcal{F}) \in \mathbb{Z}_{\geq 0}$.

Def: The Euler characteristic of $\mathcal{F}$ is

$$\chi(X,\mathcal{F}) = \sum_{i \geq 0} (-1)^i h^i(X,\mathcal{F}) \in \mathbb{Z}$$

(Aside: If X is dim d, X can be covered by $d+1$ affine opens, so $h^i(X,\mathcal{F}) = 0$ for $i > d$.)

Motivation 1 (for $\chi(X,\mathcal{F})$): Recall that we computed

$$h^i(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} \binom{n+d}{n} & \text{if } i=0,\ d \geq 0 \\ \binom{-1-d}{n} & \text{if } i=n,\ d \leq -n-1 \\ 0 & \text{else} \end{cases}$$

Then $\chi(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{n}$ is a poly in d for all $d \in \mathbb{Z}$.

Special case of something more general (Thursday):

$$\chi(\mathbb{P}^n, \mathcal{F}(d)) = \text{poly}(d) \qquad \text{"Hilbert polynomial".}$$

Motivation 2: χ is also much easier to compute than h^i, for the following reason:

Lemma: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact, then
$$\chi(X,\mathcal{G}) = \chi(X,\mathcal{F}) + \chi(X,\mathcal{H}).$$

Pf: Long exact sequence + algebra 'ct that alternating sum of ranks vanishes in an exact sequence of vector spaces.

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{H}) \arrow[dll] \\ & H^1(X,\mathcal{F}) \arrow[r] & \cdots \end{tikzcd}$$

$\blacksquare$.

Lemma: Suppose $D \subset X$ is an irred divisor that is effective Cartier, i.e. $\mathcal{I}_D \cong \mathcal{O}_X(-D)$ is a line bundle. Let $\mathcal{F}$ be a vector bundle on X.

Then.

$$\chi(X, \mathcal{F}) - \chi(X, \underbrace{\mathcal{F}(-D)}_{\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)}) = \chi(D, \mathcal{F}|_D).$$

$\boxed{\iota : D \hookrightarrow X}$

Pf: There is a short exact sequence:

$$0 \to \mathcal{I}_D \to \mathcal{O}_X \to \iota_* \mathcal{O}_D \to 0$$

Tensoring with $\mathcal{F}$ is exact and gives

$$0 \to \mathcal{F}(-D) \to \mathcal{F} \to \mathcal{F} \otimes \iota_* \mathcal{O}_D \to 0$$

$\cong$ proj. formula

$$\iota_*(\iota^* \mathcal{F} \otimes_{\mathcal{O}_D} \mathcal{O}_D) = \iota_* \iota^* \mathcal{F}$$

So $\chi(X, \mathcal{F}) - \chi(X, \mathcal{F}(-D)) = \chi(X, \iota_* \iota^* \mathcal{F})$

$= \chi(D, \iota^* \mathcal{F})$ ▨.

Cor: Suppose C is a regular projective curve over k, and suppose $\mathcal{L}$ is a line bundle on C, and suppose $p \in C$ is a closed point. Then

$$\chi(X, \mathcal{L}(p)) - \chi(X, \mathcal{L}) = \dim_k k_p .$$

Pf: $\mathcal{L}(p)\big|_p = \widetilde{k_p}$ (as a sheaf on $p = \operatorname{Spec} k_p$)

Def 1: Let $\mathcal{L}$ be a line bundle on a regular proj curve C/k. Then the *degree* of $\mathcal{L}$ is

$$\deg \mathcal{L} = \deg_C \mathcal{L} := \chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C).$$

Def 2: Let $D = \sum a_p [p] \in \operatorname{Weil} C$, C/k a regular proj. curve. Then the *degree* of D is

$$\deg D := \sum a_p (\dim_k k_p).$$

Cor: $\deg \mathcal{O}_C(D) = \deg D$. In particular, if f is a rational function on C, then $\deg(\operatorname{div}(f)) = 0$.

In other words, $\deg : \operatorname{Pic}(C) \to \mathbb{Z}$ is a group homomorphism.
(Warning: proj of C used everywhere here).

So we have

$$\chi(C,\mathcal{L}) = \underbrace{\deg \mathcal{L}}_{\substack{\text{can be read off}\\ \text{from Weil divisor}}} + \underbrace{\chi(C,\mathcal{O}_C)}_{\substack{\text{only depends on } C,\\ \text{not on } \mathcal{L}.}}$$

<u>Def</u>: Let X be a proj. k-scheme. The <u>arithmetic genus</u> of X is

$$p_a(X) := 1 - \chi(X,\mathcal{O}_X).$$

<u>This</u> is an $\mathbb{Z}$-valued invariant of proj.' schemes!

<u>Motivation 1 (for "$1-\chi$")</u>: If C is an integral curve and $k=\bar{k}$, then we know $h^0(C,\mathcal{O}_C)=1$.

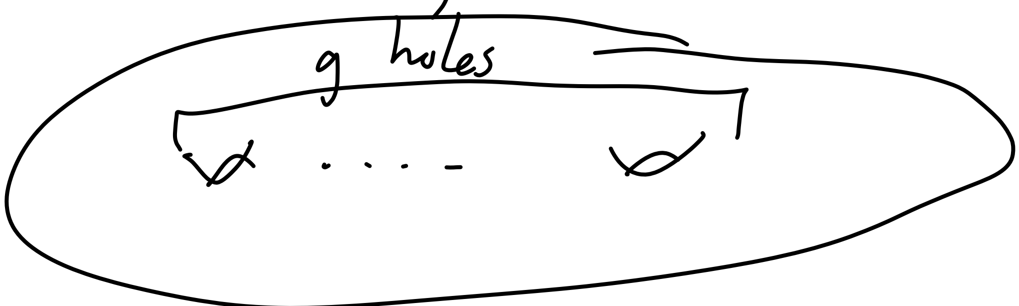
Since $\dim C < 2$, $h^i(C,\mathcal{O}_C)=0$ for $i\geq 2$.

So $p_a(C) = 1-(h^0(C,\mathcal{O}_C)-h^1(C,\mathcal{O}_C)) = h^1(C,\mathcal{O}_C) \geq 0$.

In this case (integral curve with $k=\bar{k}$) we usually write $g = p_a(C)$. ("genus of C")

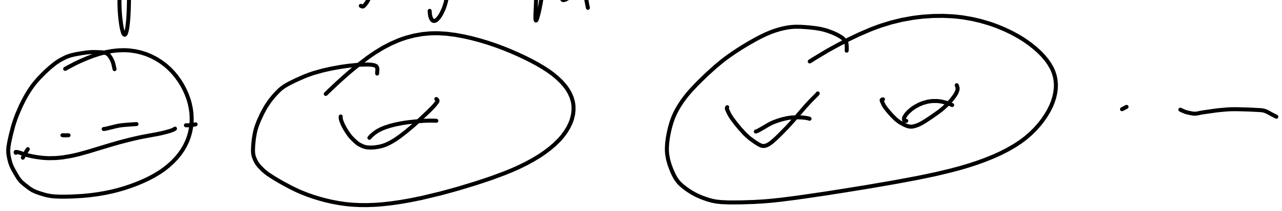
Motivation 2:

Miracle: if $k=\mathbb{C}$, a regular integral proj. curve of g looks like ($\mathbb{C}$-analytically)



as a Riemann surface!

In particular, $g=p_a(C)$ takes on values $0,1,2,\dots$



Warning: $\chi_{top}(C)=2-2g$, but $\chi(C,\mathcal{O}_C)=1-g$.

$$\chi_{top}(C) = \chi(C_{\text{analytic}}, \underline{\mathbb{Z}})$$

We can now write (C a regular integral curve over $k=\bar{k}$)

$$h^0(C,\mathcal{L})-h^1(C,\mathcal{L})=\chi(C,\mathcal{L})=\deg\mathcal{L}+1-g(C)$$

When can we understand h^0 and h^1 individually?

Interesting in general, but we can do two special cases now:

<u>Lemma</u>: If $\deg \mathcal{L} = d < 0$, then
$$h^0(C,\mathcal{L}) = 0, \ h^1(C,\mathcal{L}) = g-d-1.$$

<u>Pf</u>: Any nonzero rational section s of $\mathcal{L}$ has
$$\deg(\mathrm{div}(s)) = \deg(\mathcal{O}(\mathrm{div}\, s)) = \deg \mathcal{L} = d < 0,$$
so we cannot have $\mathrm{div}(s) \geq 0$, i.e. s is not a regular section. So $H^0(C,\mathcal{L}) = 0$. $\boxtimes$

<u>Lemma</u>: If $\deg \mathcal{L} = 0$ and $\mathcal{L} \not\cong \mathcal{O}_C$, then
$$h^0(C,\mathcal{L}) = 0, \ h^1(C,\mathcal{L}) = g-1.$$

So we understand $h^i(C,\mathcal{L})$ for $\deg \mathcal{L} \leq 0$.
What about $\mathcal{L}$ of very high degree?

A related result!

<u>Thm</u> (Serre vanishing): Let $\mathcal{F}$ be a finite type qcoh sheaf on a proj k-scheme X. Let $\mathcal{O}(1)$ be the very ample line bundle on X corresp. to some embedding in proj. space. Then for all suff large m,

$$h^i(X, \mathcal{F}(m)) = 0 \text{ for all } i > 0.$$

<u>Pf</u>: Same as argument that $h^i(X, \mathcal{F}) < \infty$, but now using input

$$h^i(\mathbb{P}^n_k, \mathcal{O}(m)) = 0 \text{ for } i > 0, m > -n-1.$$

Since a very ample line bundle on a curve must have positive degree, this suggests that $h^1(C, \mathcal{L}) = 0$ for $\deg \mathcal{L} \gg 0$.

(We prev. saw $h^0(C, \mathcal{L}) = 0$ for $\deg \mathcal{L} \ll 0$).

Thm (Serre duality, hard): Suppose X is geom. integral smooth proj. k-variety of dim n. Then there is a line bundle $\boxed{\omega_X}$ on X s.t. there are canonical $H^i(X, \mathcal{F}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{F}^\vee)^\vee$ for all $i \in \mathbb{Z}$ and vector bundles $\mathcal{F}$.

In particular, $h^i(X, \mathcal{F}) = h^{n-i}(X, \omega_X \otimes \mathcal{F}^\vee)$.

(The line bundle ω_X is called the dualizing sheaf of X.)

At end of class I'll outline a proof in the special case $\dim X = \operatorname{rk} \mathcal{F} = 1$, i.e. line bundles on curves. This special case is often called Riemann-Roch.

(First step in Serre duality: define ω_X, then check $h^n(X, \omega_X) = 1$ (dual to $h^0(X, \mathcal{O}_X)$)).

Assuming Riemann-Roch, can make a table

	$d<0$				$d > \deg \omega_C = 2g-2$
$d = \deg \mathcal{L}$	$\dots -3\ -2\ -1$	0	$1 \dots 2g-3$	$2g-2$	$2g-1, 2g, 2g+1$
$h^0(C,\mathcal{L})$	0		?		$d-g+1$
$h^1(C,\mathcal{L})$	$g-d-1$		?		0

red arrows: duality from Riemann-Roch.

interesting range: h^0, h^1 depend heavily on $C, \mathcal{L}$, not just g and d.