C = geom. regular, geom. integral proj curve $/k$

Last time: plan for studying curves:

$C \rightsquigarrow g = H^1(C, \mathcal{O}_C) = H^0(C, \omega_C)$

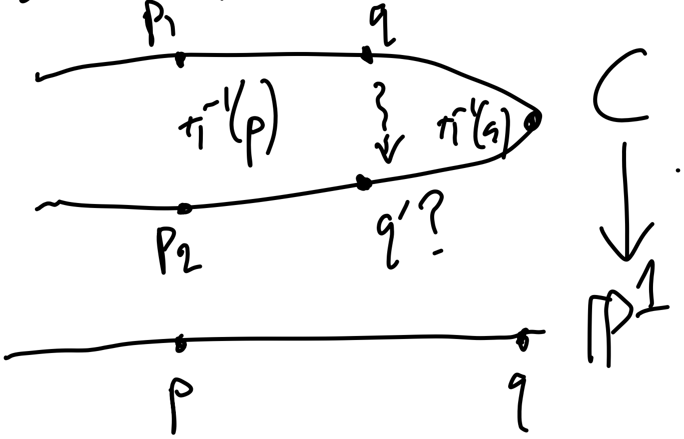
R-R $\rightsquigarrow$ deg $2g-2$ line bundle ω_C on C

if $g=0$: $|\omega_C^\vee| : C \hookrightarrow \mathbb{P}^2_k$ (conic, deg 2) ✓

if $g=1$: Can check: $\omega_C \cong \mathcal{O}_C$, so ??? (postponing)

if $g \geq 2$: $|\omega_C| : C \longrightarrow \mathbb{P}^{g-1}_k$ "canonical morphism". (today).

$g=2$: $|\omega_C| : C \longrightarrow \mathbb{P}^1_k$ is a deg $2g-2=2$ finite morphism.



Notes: ($\pi = |\omega_C| : C \longrightarrow \mathbb{P}^1_k$, $g(C) = 2$, $\deg \pi = 2$.)

1) Suppose $p \in \mathbb{P}^1_k$ has $k_p = k$ and $\pi^{-1}(p) = \{p_1, p_2\}$

Then $k_{p_1} = k_{p_2} = k$.

What does it mean to say $\pi(p_1) = \pi(p_2)$?

This means every global section of ω_C has the "same value" at p_1 and p_2. In particular,

$s(p_1) = 0 \Longleftrightarrow s(p_2) = 0$. $(s \in H^0(C, \omega_C))$.

But $H^0(C, \omega_C) \cong k^2$, so there exists a nonzero global section of ω_C vanishing at p_1.

So there exists $s \in H^0(C, \omega_C)$ with $s \neq 0$ and $s(p_1) = s(p_2) = 0$. So $\operatorname{div}(s) \geq [p_1] + [p_2]$.

But $\operatorname{div}(s)$ has deg 2, so $\operatorname{div}(s) = [p_1] + [p_2]$.

Then $\omega_C \cong \mathcal{O}_C(p_1 + p_2)$.

2) (p as in (1)):

Take a section $x \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ with $\operatorname{div}(x) = [p]$. Then we have a section

$$\pi^* x \in H^0(C, \underbrace{\pi^* \mathcal{O}_{\mathbb{P}^1}(1)}_{\omega_C}).$$

We have

$$\operatorname{div}(\pi^* x) \geq [p_1] + [p_2] \text{ as before } ((\pi^* x)(p_i) = 0),$$

so again $\operatorname{div}(\pi^* x) = [p_1] + [p_2]$ and thus

$$\omega_C \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_C(\underbrace{p_1 + p_2}_{\text{fiber above } p}).$$

3) Suppose $q \in C$ with $k_q = k$. Then R-R says

$$h^0(C, \omega_C(-q)) = h^0(C, \mathcal{O}_C(q)) = 1 > 0.$$

So there is some nonzero global section

$$s \in H^0(C, \underbrace{\omega_C(-q)}_{\text{deg 1 bundle}}).$$

Then $\operatorname{div}(s)$ is a deg 1 effective Weil divisor, i.e. $\operatorname{div}(s) = [q']$ for some q' in C with $k_{q'} = k$.

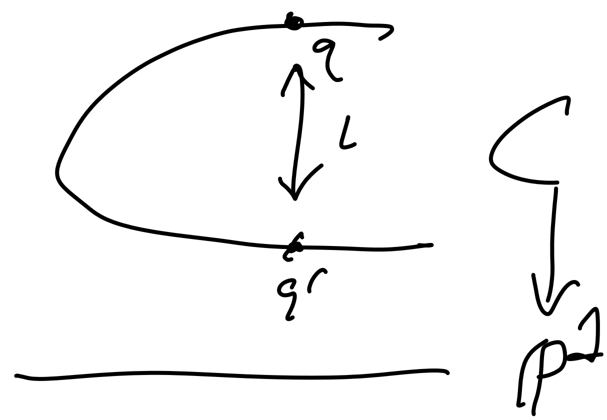
Then $\omega_C(-q) \simeq \mathcal{O}_C(q')$.

Can use this to define a morphism

$$\iota : \underbrace{C(k)}_{k\text{-valued points}} \longrightarrow C(k)$$

$$q \longmapsto q'$$

This will be an involution



Really ι comes from an order 2 automorphism of C, the <u>hyperelliptic involution</u>.

<u>Def</u>: A curve C is <u>hyperelliptic</u> if it admits a degree 2 map $C \longrightarrow \mathbb{P}^1_k$.

<u>Note</u>: Equivalently ($g > 0$), there exists a deg 2 line bundle $\mathcal{L}$ on C with $h^0(C, \mathcal{L}) = 2$.

We've seen: every genus 2 curve is hyperelliptic.

Let's postpone further discussion of hyperelliptic curves just like we did the case $g = 1$.

__Thm:__ Let $g(C) \geq 2$. Then exactly one of the following is true:

- C is hyperelliptic
- the canonical morphism $|\omega_C|: C \to \mathbb{P}_k^{g-1}$ is a closed embedding. ("canonical embedding").

__Pf:__ Assume $k = \bar{k}$ for simplicity.

Recall ω_C is very ample $\Longleftrightarrow$

$$h^0(C, \omega_C(-p-q)) = h^0(C, \omega_C(-p)) - 1$$

for all $p, q \in C$.

R-R: $h^0(C, \omega_C) = g$

$\deg(\omega_C(-p-q)) = 2g-4$

$h^0(C, \omega_C(-p)) = g-1$

$$h^0(C, \omega_C(-p-q)) = \underbrace{h^0(C, \mathcal{O}_C(p+q))}_{\substack{1 \text{ or } 2, \\ 2 \text{ for some } p, q \\ \Longleftrightarrow C \text{ is hyperelliptic}}} + \overbrace{d}^{} - g + 1$$

$= g - 3 + [1 \text{ or } 2]$

$\blacksquare$.

$\underline{g=3:}$

Suppose C is a non-hyperelliptic curve of genus 3.

Then $|\omega_C| : C \hookrightarrow \mathbb{P}^2_k$ identifies C with a plane curve of degree $2\cdot 3-2=4$.

This identification is $\underline{\text{canonical}}$ up to a choice of basis for $H^0(C,\omega_C)$, i.e. up to the action of PGL_3 on $\mathbb{P}^2_k$.

$\underline{\text{Thm}}$:

$$\{\text{genus 3 curves}\}/\text{iso} = \left\{\begin{matrix}\text{hyperell. genus}\\ \text{3 curves}\end{matrix}\right\}/\text{iso} \sqcup \left\{\begin{matrix}\text{regular}\\ \text{plane}\\ \text{quartics}\end{matrix}\right\}/PGL_3.$$

$\underline{\text{Pf}}$: Observations that are needed:

1) Any regular plane quartic is indeed genus 3 ($\binom{d-1}{2}$ formula).

2) If $\pi : C \hookrightarrow \mathbb{P}^2_k$ is a regular plane quartic, then $\pi^*\mathcal{O}_{\mathbb{P}^2_k}(1) \cong \omega_C$. Why?

$h^0(C,\pi^*\mathcal{O}(1)) \geq h^0(\mathbb{P}^2,\mathcal{O}(1)) = 3$, so (R-R)

$h^0(C,\omega_C \otimes (\pi^*\mathcal{O}(1))^\vee) \geq 1$, but $\deg(\omega_C \otimes (\pi^*\mathcal{O}(1))^\vee) = 0$,

so this means $\omega_C \otimes (\pi^*\mathcal{O}(1))^\vee \cong \mathcal{O}_C$. $\blacksquare$

<u>$g=4$!</u>

(again assume C non-hyperelliptic)

Now have $C \hookrightarrow \mathbb{P}^3_k$ is a degree 6 curve in 3-space.

<u>Not</u> a hypersurface, need to know more.

One type of degree 6 curve in $\mathbb{P}^3$:

intersection of surfaces of degrees 2 and 3

(Bezout's Thm).

Hilbert poly calc (pset): such an intersection indeed has genus 4.

<u>Goal:</u> show that $|\omega_C|$ identifies C with such a "2,3-complete intersection".

Q: Which deg d hypersurfaces contain the image of $|w_C|$?

Consider the composition

Veronese embedding

$$\begin{tikzcd} C \arrow[r, hook, "|w_C|"] \arrow[rr, bend right, "\pi"'] & \mathbb{P}^3_k \arrow[r, hook, "v_d"] & \mathbb{P}_k^{\binom{d+3}{3}-1} \end{tikzcd}$$

The deg d hypersurfaces in $\mathbb{P}^3$ correspond to hyperplanes in $\mathbb{P}^{\binom{d+3}{3}-1}$.

Want to know: which hyperplanes contain $\pi(C)$?

Let $\alpha : k^{\binom{d+3}{3}} \longrightarrow H^0(C, \pi^*\mathcal{O}(1))$ be the linear system corresponding to $\pi : C \hookrightarrow \mathbb{P}_k^{\binom{d+3}{3}-1}$.

Hyperplanes containing $\pi(C) \longleftrightarrow \ker \alpha$.

Note that
$$\pi^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq |w_C|^* v_d^* \mathcal{O}_{\mathbb{P}^N}(1)$$
$$= |w_C|^* \mathcal{O}_{\mathbb{P}^3}(d)$$
$$= \left(|w_C|^* \mathcal{O}_{\mathbb{P}^3}(1)\right)^{\otimes d} = w_C^{\otimes d}$$

So $\deg(\pi^* \mathcal{O}(1)) = d \cdot \deg(\omega_C) = 6d$,
and then Riemann-Roch gives ($d > 1$)

$$h^0(C, \pi^* \mathcal{O}(1)) = 6d - 4 + 1 = 6d - 3.$$

Thus

$\{ f$ homog poly of deg d in $x_0, x_1, x_2, x_3 \mid C \subseteq V(f) \}$ is a vector space

of dim $\geq \binom{d+3}{3} - (6d-3)$.

$d=2$: get dim ≥ 1
$d=3$: get dim ≥ 5

So for any canonically embedded genus 4 curve $C \hookrightarrow \mathbb{P}^3$, there exists nonzero f of deg 2 and a g of deg 3 not a multiple of f

with $V(f), V(g) \supseteq C$.

So $C \subseteq V(f) \cap V(g)$. Equal because both sides have the same Hilbert polynomial.

Thm: $\{C \mid g(C)=4\}/_{\text{iso}} = \{\underset{C}{\text{hyperelliptic}} \mid g(C)=4\}/_{\text{iso}}$

$\sqcup \{C \ (2,3) \text{ complete intersections in } \mathbb{P}^3\} / PGL_4$

$g=5$: $C \hookrightarrow \mathbb{P}^4$ deg 8 curve.

Can check: a $(2,2,2)$ complete intersection has genus 5,

Can check: $C \subseteq V(f) \cap V(g) \cap V(h)$ for f, g, h linearly independent deg 2 polys.

Problem: $V(f) \cap V(g) \cap V(h)$ might still have dimension 2, not a curve.

Harder:

Thm: $\left\{\begin{matrix}\text{genus 5} \\ \text{curves}\end{matrix}\right\} = \{\text{hyperelliptic}\} \sqcup \{(2,2,2)\text{-c.i. in } \mathbb{P}^4\}$

$\sqcup \{\underbrace{\text{trigonal}}_{\text{triple covers of } \mathbb{P}^1}\}$