Last time: • C is hyperelliptic $\overset{\text{def}}{\Longleftrightarrow}$ C is a double cover of $\mathbb{P}^1$

• description of all non-hyperelliptic curves of genus g for $2 \le g \le 5$.

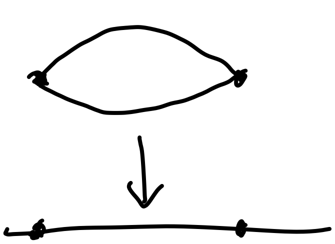
Today: Suppose C is an integral regular projective curve over $k = \bar{k}$, char $k \neq 2$.

Goal: parametrize all genus g hyperelliptic curves.

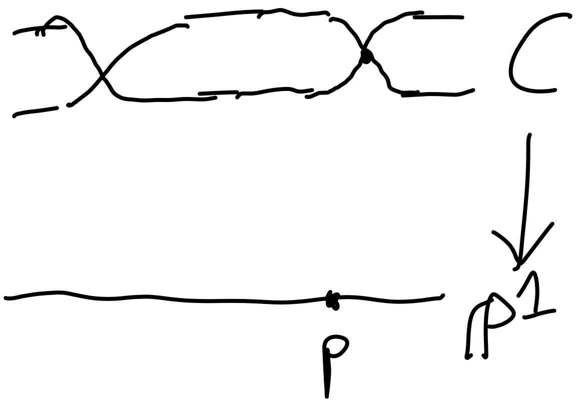
Def: Suppose $\pi: C \to \mathbb{P}^1_k$ is a degree 2 map. Then a closed point $p \in \mathbb{P}^1_k$ is a branch point of π if $\# \pi^{-1}(p) = 1$, and a closed point $q \in C$ is a ramification point of π if $\pi(q)$ is a branch point.

Example: $\pi: \mathbb{P}^1_k \to \mathbb{P}^1_k$ is branched over $[1:0]$ and $[0:1]$

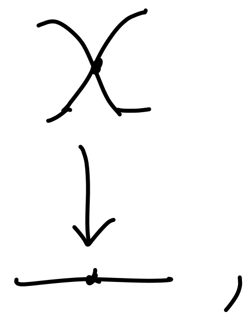
$$[x:y] \longmapsto [x^2:y^2]$$



General picture:



possible confusion: C is still regular at points



this picture should be viewed $(/\mathbb{C})$ as representing

$$\begin{array}{c}\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}\\ \big\downarrow z^2\\ \mathbb{D}\end{array}$$

<u>Thm</u>: Suppose $k=\bar{k}$, char $k\neq 2$, $r>0$, and $p_1,\dots,p_r$ are distinct closed points in $\mathbb{P}^1_k$.
If r is odd, there is no double cover of $\mathbb{P}^1_k$ branched precisely over $p_1,\dots,p_r$. If r is even, there is a unique such double cover $C\to\mathbb{P}^1_k$, and C has genus $\frac{r}{2}-1$.

<u>Q</u>: How do we find these double covers?

A: function fields: $C\to\mathbb{P}^1_k \longleftrightarrow K(\mathbb{P}^1_k)\hookrightarrow \underset{\text{quad. extension}}{K(C)}$

Lemma: Suppose $f = (x-\alpha_1)\cdots(x-\alpha_{2m}) \in k[t]$ is a monic polynomial with distinct nonzero roots $\alpha_1, \dots, \alpha_{2m} \in k$. Then the (<u>unique</u>) double cover $\pi : C \to \mathbb{P}^1_k$ corresponding to the extension of function fields

$$k(x) \hookrightarrow k(x)(\sqrt{f(x)})$$

is given by gluing

$$\underbrace{\operatorname{Spec} k[x,y]/(y^2 - f(x))}_{\cong\, k[x][\sqrt{f(x)}]} \longrightarrow \operatorname{Spec} k[x] \cong \mathbb{A}^1_k \quad \text{to}$$

$$\operatorname{Spec} k[u,v]/(v^2 - \underbrace{u^{2m} f(\tfrac{1}{u})}_{g(u)}) \longrightarrow \operatorname{Spec} k[u] \cong \mathbb{A}^1_k$$

over

$$\begin{tikzcd}
\operatorname{Spec} k[x,x^{-1},y]/(y^2 - f(x)) \arrow[r] \arrow[d, "\cong"] & \operatorname{Spec} k[x,x^{-1}] \arrow[d, "\cong"] \\
\operatorname{Spec} k[u,u^{-1},v]/(v^2 - u^{2m} f(\tfrac{1}{u})) \arrow[r] & \operatorname{Spec} k[u,u^{-1}]
\end{tikzcd}
\qquad
\boxed{\begin{array}{cc} 1/x & y/x^m \\ \uparrow & \uparrow \\ u & v \end{array}}$$

Pf: 1) Check gluing makes sense. Then we have a morphism $\pi : C \to \mathbb{P}^1_k$.

2) Check C is integral, $K(\mathbb{P}^1_k) \to K(C)$ is what we wanted.

3) C is proj$/k$ because $\pi : C \to \mathbb{P}^1_k$ is finite, hence proj, so $C \to \mathbb{P}^1_k \to \operatorname{Spec} k$ is proj.

4) C is regular because $\operatorname{Spec} k[x,y]/y^2 - f(x)$ is (uses that f is squarefree), and similarly on other chart. $\boxtimes$

Pf of Thm:

Any double cover of $\mathbb{P}^1_k$ corresponds to some extension

$$k(x)(\sqrt{f(x)}) \text{ for some } f \in k(x)^* / (k(x)^*)^2 - \{1\}.$$

By multiplying by a square and applying an autom

$$k(x) \to k(x), \quad x \mapsto \frac{ax+b}{cx+d}$$

if necessary, can assume f is as in the lemma. So $\pi: C \to \mathbb{P}^1_k$ is as constructed in the lemma.

Examination: π is branched over precisely the $2m$ points in the first chart corresp. to the roots of f.

$$\begin{array}{ccc} V(y^2 - f(x)) & & [x,y] \\ \downarrow & & \downarrow \\ \mathbb{A}^1_k & & x \end{array}$$

(in particular, π is not branched over "∞").

Conclusion: Every double cover of $\mathbb{P}^1_k$ is branched over an even number of points, and for any given set of an even number of branch points, there is exactly one such double cover.

Remains to show: $g(C) = \frac{r}{2} - 1 = m - 1$

Two approaches to the genus computation:

1) Compare ω_C and $\pi_1^* \omega_{\mathbb{P}^1}$ to compute $\deg(\omega_C) = 2g-2$.
   – we'll do this after constructed ω_C.

2) Compute $H^1(C, \mathcal{O}_C) = g$ by Čech cohom on given affine open cover.

$H^1(C, \mathcal{O}_C)$ is the cokernel of the map

$$
\begin{array}{ccc}
k[x,y]/(y^2 - f(x)) & \oplus & k[u,v]/(v^2 - u^{2m} f(\tfrac{1}{u})) \\
\big\downarrow & & u \mapsto x^{-1} \quad v \mapsto x^{-m} y. \\
k[x, \tfrac{1}{x}, y]/(y^2 - f(x)) & &
\end{array}
$$

Can check: $H^1(C, \mathcal{O}_C)$ has basis $x^{-1}y, x^{-2}y, \ldots, x^{1-m}y$,

so $h^1(C, \mathcal{O}_C) = m - 1 = \frac{r}{2} - 1$. $\blacksquare$.

Two loose ends:

Q1: Can a given curve C admit multiple deg 2 maps $C \to \mathbb{P}^1_k$ (i.e. not related by composition with an aut of $\mathbb{P}^1_k$)

Q2: If C is hyperelliptic, of genus ≥ 1 we know that $|\omega_C| : C \to \mathbb{P}^{g-1}$ is not a closed embedding. What is this morphism?

Prop: Let $\pi : C \to \mathbb{P}^1_k$ is deg 2 and let $g(C) \geq 2$. Then the canonical morphism is the composition

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[rr, bend right, "\alpha"'] & \mathbb{P}^1 \arrow[r, hook, "v_{g-1}"'] & \mathbb{P}^{g-1} \end{tikzcd}$$

$\leftarrow$ Veronese.

Pf: Let α be the composition and let $\mathcal{L} = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$, so $\deg \mathcal{L} = 2$, $h^0(C, \mathcal{L}) = 2$.

Then $\alpha^* \mathcal{O}_{\mathbb{P}^{g-1}}(1) = \pi^* \mathcal{O}_{\mathbb{P}^1}(g-1) = \mathcal{L}^{\otimes(g-1)}$ is a deg $2g-2$ line bundle. Since the image of v_{g-1} is not contained in a hyperplane, $h^0(C, \mathcal{L}^{\otimes(g-1)}) \geq g$.

Then R-R gives

$$h^0(C, \omega_C \otimes (\mathcal{L}^{\otimes(g-1)})^\vee) \geq 1,$$

so $\omega_C \otimes (\mathcal{L}^{\otimes(g-1)})^\vee \cong \mathcal{O}_C$.

So $\alpha^* \mathcal{O}_{\mathbb{P}^{g-1}}(1) \cong \omega_C$, so

$$h^0(C, \alpha^* \mathcal{O}_{\mathbb{P}^{g-1}}(1)) = h^0(C, \omega_C) = g$$

and α must correspond to a complete linear system, i.e. $\alpha = |\omega_C|$. ▨

<u>Cor</u>: If C is a hyperelliptic curve of genus ≥ 2, the hyperelliptic morphism $C \to \mathbb{P}^1$ is unique.

<u>Cor</u>: Let $g \geq 2$ and suppose $k = \bar{k}$, char $k \neq 2$. Then there is a bijection

$$\left\{ \begin{array}{c} \text{hyperelliptic curves} \\ \text{of genus } g \end{array} \right\} \Big/ \text{iso} \xleftrightarrow{\sim} \left\{ \begin{array}{c} S \subset \mathbb{P}^1_k - \{2\}, \\ |S| = 2g+2 \end{array} \right\} \Big/ \underset{\text{``}PGL_2(k)\text{''}}{\mathrm{Aut}(\mathbb{P}^1_k)}$$

"dim $2g+2-3$
$= 2g-1$"

genus	type of curve	"dimension" (number of parameters)
2	hyperelliptic	3
3	hyperelliptic	5
	plane quartic	$6 = h^0(\mathbb{P}^2, \mathcal{O}(4)) - 1 - \dim PGL_3$
4	hyperelliptic	7
	(2,3) complete intersection	$9 = h^0(\mathbb{P}^3, \mathcal{O}(2)) - 1 + h^0(\mathbb{P}^3, \mathcal{O}(3)) - 1 - \dim PGL_4 - h^0(\mathbb{P}^3, \mathcal{O}(1))$
5	hyperelliptic	9
	trigonal	11 (14 branch points)
	(2,2,2) complete intersection	$12 = 3(h^0(\mathbb{P}^4, \mathcal{O}(2)) - 3) - \dim PGL_5$
⋮		
9	hyperelliptic	$2g-1$
	non-hyperelliptic	$3g-3$ (hard)