Today:
- a little more about families of curves of $g \geq 3$ (mainly $g = 3$)
- curves of genus 1

Next week: no class Tuesday (university-wide)

Thursday:
- info on final paper/oral exam
- start in with Ch. 21 (skipping 20), differentials.

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genus	type of curve	"dimension" (number of parameters)
2	hyperelliptic	3
$\boxed{3}$	hyperelliptic	5
	plane quartic	$6 = h^0(\mathbb{P}^2, \mathcal{O}(4)) - 1 - \dim PGL_3$
4	hyperelliptic	7
	(2,3) complete intersection	$9 = h^0(\mathbb{P}^3, \mathcal{O}(2)) - 1 + h^0(\mathbb{P}^3, \mathcal{O}(3)) - 1 - \dim PGL_4 - h^0(\mathbb{P}^3, \mathcal{O}(1))$
5	hyperelliptic	9
	trigonal	11 (14 branchpoints)
	(2,2,2) complete intersection	$12 = 3(h^0(\mathbb{P}^4, \mathcal{O}(2)) - 3) - \dim PGL_5$
⋮		
9	hyperelliptic	$2g-1$
	non-hyperelliptic	$3g-3$ (hard)

Fact: For $g = 3, 4, 5, \ldots$ we shouldn't think of

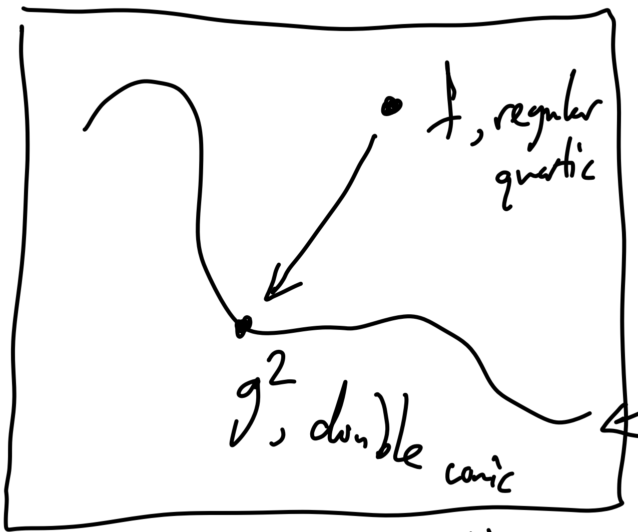
$\{\text{genus } g \text{ curves}\}/\text{iso}$ as having multiple irred components of diff dimensions, but rather that the different types of curves fit together in a single irred family.

"M_g = "moduli of genus curves" is irreducible".

Example: $g=3$: can interpret hyperelliptic curves as "limits" of quartic plane curves, as follows.

(Motivation: canonical morphism for hyperelliptic genus 3 curve is

$C \longrightarrow \mathbb{P}^2$, double cover of a conic
"conic with multiplicity 2")



$$\underbrace{\mathbb{P}^{14}}_{\text{quartic polys in } x_0, x_1, x_2} / \underbrace{PGL_3}_{\mathrm{Aut}(\mathbb{P}^2)}$$

divisor of singular plane quartics

(i.e. $V(g)$ = regular conic,
$V(g^2)$ = non-reduced "double conic")

What is the "limit" as we approach $V(g^2)$ from the direction of $V(f)$,
i.e. by $\lim\limits_{t\to 0} V(g^2+tf)$?

Genus 4: similar but more complicated story
(hyperelliptic locus is codim 2, not 1)

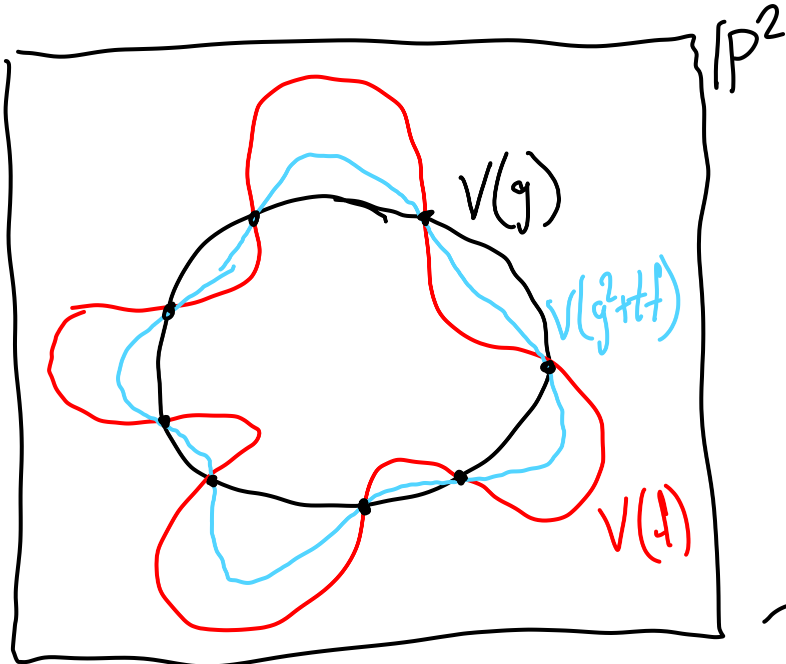
Genus 1 curves:

Difficulties if $k \neq \bar{k}$: issue is that for any $n > 0$, there may exist C with no line bundles of degrees $1, 2, \ldots, n$ (and thus also no closed points of degree $\leq n$).

Point: the dualizing sheaf $\omega_C \cong \mathcal{O}_C$ is useless for this.

What if we are given a line bundle $\mathcal{L}$ of degree $d > 0$?

R-R: $h^0(C, \mathcal{L}) = d$, $\mathcal{L}$ is base-point-free for $d \geq 2$,
$\mathcal{L}$ is very ample for $d \geq 3$.



2.4 (Bezout's Thm)
$\shortparallel$
8 base points for this pencil of curves.

So the choice of $V(f)$ determines 8 points on $V(g) \cong \mathbb{P}^1_k$

$\rightsquigarrow$ corresponds to the hyperelliptic curve branched over those 8 points.

Claim: $\displaystyle \text{"}\lim_{t\to 0}\text{"}\, \underbrace{V(g^2+tf)}_{\text{regular quartic plane curve}} = \underbrace{\text{double cover of } V(g) \text{ branched over } V(g)\cap V(f)}_{\text{genus 3 hyperelliptic}}.$

More rigorous description of what is happening:

$$\begin{tikzcd}
\pi^{-1}(\{0\}) \arrow[r, hook] \arrow[d] & X \arrow[r, "\text{normalization}"] \arrow[d, "\pi"] & V(g^2+u^2f) \arrow[r] \arrow[d] & V(g^2+tf) \subset \mathbb{A}^1\times\mathbb{P}^2 \arrow[d] \\
\{0\} \arrow[r, hook] & \mathbb{A}^1 = \operatorname{Spec} k[u] \arrow[r, equal] & \mathbb{A}^1 \arrow[r] & \mathbb{A}^1 = \operatorname{Spec} k[t]
\end{tikzcd}$$

$u^2 \mapsfrom t$

$\pi^{-1}(\{0\})$ = hyperelliptic genus 3 curve.

$V(g^2+u^2f)$: singular surface along $\{u=0\}$

$V(g^2+tf)$: surface

The "limit" in question is computed by the "nice" family π.

$d=1$: $h^0(C,\mathcal{L})=1 \Rightarrow$ there's a unique nonzero section s (up to scaling)

Then $\operatorname{div}(s)=[p]$ for some deg 1 point $p\in C$.

Conclusion: $\mathcal{L}\cong \mathcal{O}(p)$ for a unique deg 1 point $p\in C$.

$d=2$: $h^0(C,\mathcal{L})=2 \Longrightarrow$ hyperelliptic morphism

$$|\mathcal{L}|: C \longrightarrow \mathbb{P}^1_k$$

By previous work, if $k=\bar{k}$, this is branched over 4 points.

<u>Note</u>: Different $\mathcal{L}$ can give different hyperelliptic morphisms.

$d=3$: Now very ample, $|\mathcal{L}|$ identifies C with a cubic plane curve.

$d=4$: space curve of degree 4.

pset: this is a $(2,2)$-complete intersection.

Def: An <u>elliptic curve</u> is a pair (E, p) where

E is a geom. integral geom. regular proj curve/k of genus 1 and $p \in E$ is a k-valued point (deg 1 point).

<u>Remark</u>: We'll see that Aut(genus 1 curve/k) acts transitively on the set of k-valued points, so this def is close to just requiring that E has at least one k-valued point.

But fixing p is still nice – one motivation is that $\mathrm{Aut}(E) = \mathrm{Aut}((E, p))$ will be finite.

So if $E = (E, p)$ is an elliptic curve, we can view E as a double cover of $\mathbb{P}^1$, cubic plane curve, $(2,2)$-c.i., etc – <u>many</u> "canonical" descriptions of E. using $\mathcal{L} = \mathcal{O}(d[p])$.

Q: What is $\mathrm{Pic}(E)$?

Suppose $\mathcal{L} \in \mathrm{Pic}(E)$ has degree $d \in \mathbb{Z}$.

Then $\mathcal{L}(-(d-1)[p])$ has degree 1, so is $\mathcal{O}_E(q)$ for some deg 1 point $q \in \underbrace{E(k)}_{\{x \in E \mid k_x = k\}}$.

Thus

$$\mathrm{Pic}(E) = \left\{ \mathcal{O}_E((d-1)[p] + [q]) \;\middle|\; \begin{matrix} d \in \mathbb{Z} \\ q \in E(k) \end{matrix} \right\}.$$

$$\cong \underbrace{\mathbb{Z}}_{\substack{\text{gen by} \\ \mathcal{O}_E(p)}} \times \underbrace{\mathrm{Pic}^0(E)}_{\ker(\deg : \mathrm{Pic}(E) \to \mathbb{Z})}$$

and we have a bijection

$$\mathrm{Pic}^0(E) \cong E(k)$$
$$\mathcal{O}([q]-[p]) \longleftrightarrow q.$$

Since $\mathrm{Pic}^0(E)$ is an abelian group, this gives $E(k)$ the structure of an abelian group, with identity element p.

Explicitly: $[\underset{\substack{\uparrow \\ \text{addition in } E(k)}}{q+r}] = [q]+[r]-[p]$ (in $Cl(E)$)

Returning to $d=2$:

We have $\pi: E \to \mathbb{P}^1_k$, $\pi^* \mathcal{O}_{\mathbb{P}^1_k}(1) \cong \mathcal{O}_E(2[p])$.

Suppose $k = \bar{k}$, so π has 4 ramif points (in E).

What are they?

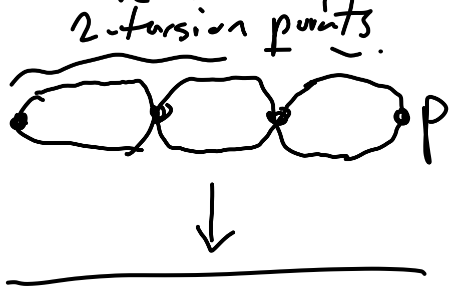
(Recall): π is ramified at $q \in E(k)$

$\iff \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_E(2[q])$,

i.e. if $2[q] = 2[p]$ in $Cl(E)$, i.e

q is a 2-torsion point.

We conclude that E has exactly 4 2-torsion points (if $k=\bar{k}$).



Similarly, the hyperelliptic involution $\iota: E \to E$

Looks like $q \mapsto \underbrace{-q}_{\text{inverse in the group } E(k)}$ on k-valued points

$d=3$ (again): $\pi : E \hookrightarrow \mathbb{P}^2_k$, $\pi^* \mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_E(3[p])$.

A line in $\mathbb{P}^2$ intersects E in 3 points (with multiplicities). What are the constraints on these points? Must have

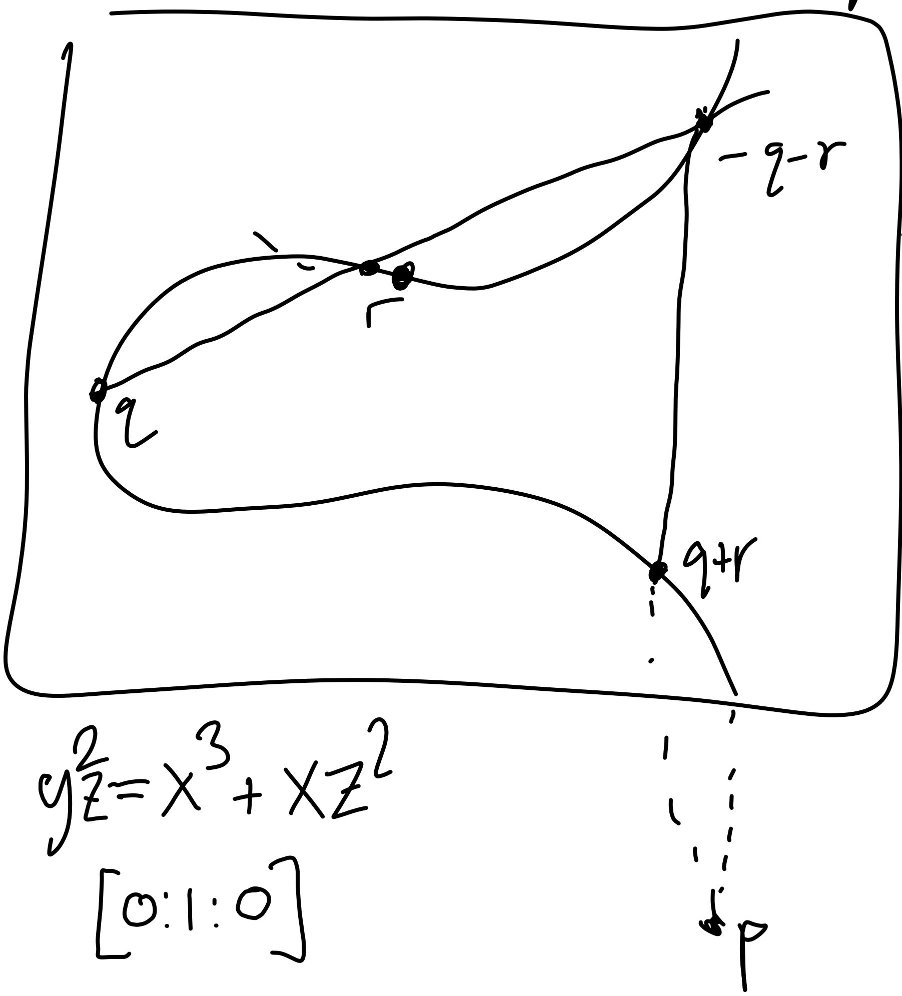
$$\underbrace{[q_1]+[q_2]+[q_3]}_{\text{"set of points where } \pi^* l \text{ vanishes"}} = \underbrace{3[p]}_{\operatorname{div}(\pi^* l),\ l \in H^0(\mathbb{P}^2, \mathcal{O}(1))} \text{ in } Cl(E)$$

$$l \cap E = \{q_1, q_2, q_3\}$$

So $q_1 + q_2 + q_3 = \underset{p}{\underset{\parallel}{0}}$ in $E(k)_{\text{group}}$

$\iff l \cap E = \{q_1, q_2, q_3\}$ for some l.

Classical description of group law on $E(k)$:



$\mathbb{P}^2$

We also have that $3[p] = 3[p]$, i.e. there should be some line intersecting E only at p, with multiplicity 3. So picture is constrained by "[illegible]ing" that p is a "flex" of $E \subseteq \mathbb{P}^2$.

$$y^2 Z = X^3 + XZ^2$$

$$[0:1:0]$$

A different choice of $p \in E(k)$ corresponds to an embedding of E in $\mathbb{P}^2$ with different flexes.