Last time:

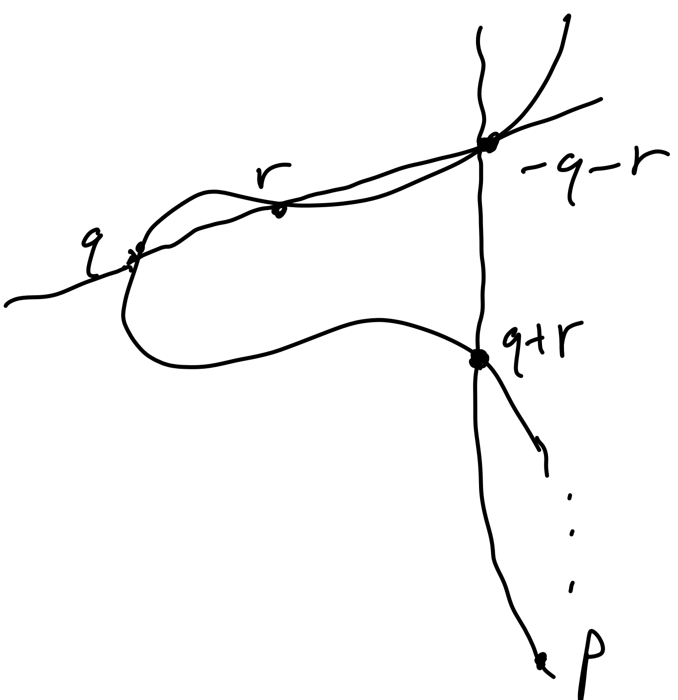
$E = (E, p \in E(k))$ elliptic curve (genus 1)

The set $E(k)$ of k-valued points of E is an abelian group via

$$E(k) \cong \mathrm{Pic}^0(E) = \{\deg 0 \text{ line bundles on } E\}/\text{iso}$$
$$q \mapsto \mathcal{O}_E([q]-[p]).$$

The line bundle $\mathcal{O}(3[p])$ identifies E with a cubic plane curve.

Then the group operations can be computed geometrically:



Thm: E is a "group scheme/k", i.e. there exist morphisms of k-schemes

$$e: \mathrm{Spec}\, k \to E \qquad (0 \in G)$$
$$i: E \to E \qquad (g \mapsto -g)$$
$$m: E \times_k E \to E \qquad ((g,h) \mapsto g+h)$$

satisfying all the axioms of a group

(
$$\begin{tikzcd} E \arrow[r, "(i, id)"] \arrow[dr] & E \times_k E \arrow[r, "m"] & E \\ & \mathrm{Spec}\, k \arrow[ur, "e"] & \end{tikzcd}$$
commutes, etc)

Pf: write out equations for $(q,r) \mapsto -q-r$ ☒

<u>Cor</u>: Suppose $q \in E(k)$. Then

$$E \xrightarrow[(q,id)]{} E \times_k E \xrightarrow[m]{} E$$ is an automorphism of

the underlying genus 1 curve E mapping p to q.

<u>Cor</u>: Suppose $k = \bar{k}$. Then

$$\{\text{genus 1 curves}\}/\text{iso} \cong \{\text{elliptic curves}\}/\text{iso}$$

$$\cong \{S \subset \mathbb{P}^1_k(k) \mid |S| = 4\}/PGL_2 .$$

$\left(\begin{array}{l}\text{double cover} \\ |\mathcal{O}_E(2p)| : E \to \mathbb{P}^1_k\end{array}\right)$

Plan for final month:

- 8 days of class (counting today)
  - ~5 days on differentials (Ch. 21)
    - one final problem set on this, posted next week
      - and due 2 weeks from today.
  - ~3 days on proving Riemann-Roch.
- final project (paper or oral exam/presentation) on a topic of your choice.
  - document with more about this on my website (below the problem sets), including sample topics.
  - due at end of classes (April 20)

Differentials:

Basic motivation: if X is regular of dim n, we think of X as a manifold of dim n, and we would like tangent and cotangent bundles on X.

Recall: if $p \in X$, the cotangent space to X at p is

$$T^\vee_{X,p} := \underbrace{m_p/m_p^2}_{\text{a } k_p\text{-vector space}} \text{ for max. ideal } m_p \subset \mathcal{O}_{X,p}$$

It's natural to ask for a coh. sheaf $T^\vee_X$ on X whose fiber at p is $T^\vee_{X,p}$.

But this isn't quite right:

$$\dim_{k_p} T^\vee_{X,p} \underbrace{=}_{X \text{ regular at } p} \operatorname{codim}_X p \text{ is not a ``good'' function of } p.$$

In particular, if η is a generic point for an integral scheme X, then $T^\vee_{X,\eta} = 0$, but we certainly don't want our cotangent sheaf to have 0 fiber at η (since then sheaf would be 0)

Two obstacles/mysteries:

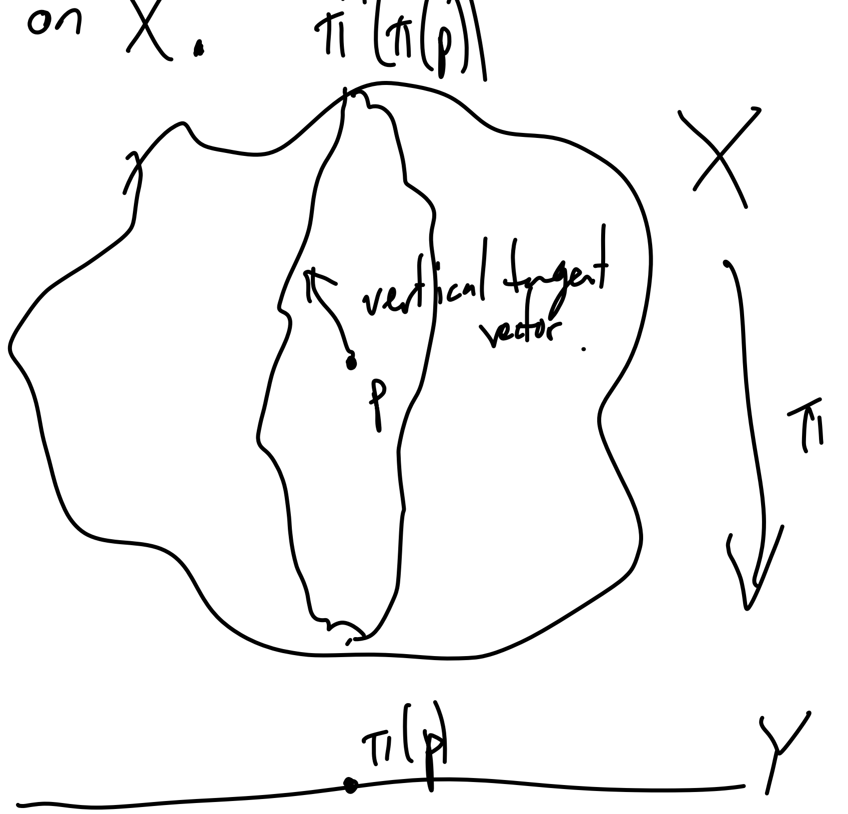
1) How do we construct a sheaf combining (some of) the vector spaces $T^\vee_{X,p}$?

2) What's going on at non-closed points?

Idea/generalization that helps us: the cotangent sheaf should really be a relative concept:
given a morphism $\pi: X \to Y$, we will define a <u>relative cotangent sheaf</u> (or <u>sheaf of relative differentials</u>)

$\Omega_\pi = \Omega_{X/Y}$ on X.

Geometric interpretation: "vertical" (in the fiber direction) (co)tangent vectors



$$\boxed{T^\vee_X := \Omega_{X/\operatorname{Spec} k} \text{ for a } k\text{-variety } X}$$

Two approaches to constructing $\Omega_{X/Y}$:

1) direct description on affines:

Suppose $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ (so A is a B-alg).

We want an A-module $\Omega_{A/B}$.

<u>Def</u>: $d : A \to \Omega_{A/B}$ is the <u>universal B-linear derivation of A</u>,

i.e. a map of B-modules satisfying

$$d(fg) = \underbrace{f \cdot d(g)}_{\text{using the } A\text{-mod structure on } \Omega_{A/B}} + g \cdot d(f) \quad \text{for all } f, g \in A,$$

such that any other such $d' : A \to M$ factors uniquely through d:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{A/B} \arrow[d, dotted, "\exists! \text{ morphism of } A\text{-modules}"] \\ & M \end{tikzcd}$$

Construction of $\Omega_{A/B}$: if $A \cong B[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$

have $\Omega_{A/B} = \bigoplus_{i=1}^{n} A \cdot dt_i \,/\, \langle \underbrace{df_1, \ldots, df_m}_{\substack{\text{expanded in terms of} \\ \text{the } dt_i \text{ using Leibniz.}}} \rangle$

Can do computations with $\Omega_{A/B}$ easily
— behaves like 1-forms in calculus.

<u>Remark</u>: alg. motivation for all of this:
if A is a k-alg and $m \subset A$ is a max. ideal
with $A/m = k$, then a
k-linear derivation $d: A \to A/m$ is the
same thing as an element of $(m/m^2)^\vee$.

"universal derivation = dual space to space of derivations"

2) Second approach to $\Omega_{X/K}$:

Idea: if $m \subset A$ is maximal, m/m^2 is the cotangent space to the closed point $\operatorname{Spec} A/m$ inside $\operatorname{Spec} A$.

What happens if we replace m with a non-maximal ideal $I \subset A$?

Hope: (true): I/I^2 is the conormal sheaf to the closed subscheme $\operatorname{Spec} A/I$ inside $\operatorname{Spec} A$.

(I/I^2 is an A/I-module, so should correspond to some qcoh sheaf on $\operatorname{Spec} A/I$)

We can extend this further: suppose $Z \hookrightarrow X$ is a closed subscheme with ideal sheaf $\mathcal{I}_Z$. Then $\mathcal{I}_Z/\mathcal{I}_Z^2$ is a qcoh sheaf on X.

With a little thought: this is actually naturally a qcoh sheaf on Z.

Def: The <u>conormal sheaf</u> of $Z \xhookrightarrow[\text{closed}]{} X$ is

$$\mathcal{N}^\vee_{Z/X} := \mathcal{I}_Z / \mathcal{I}_Z^2 .$$

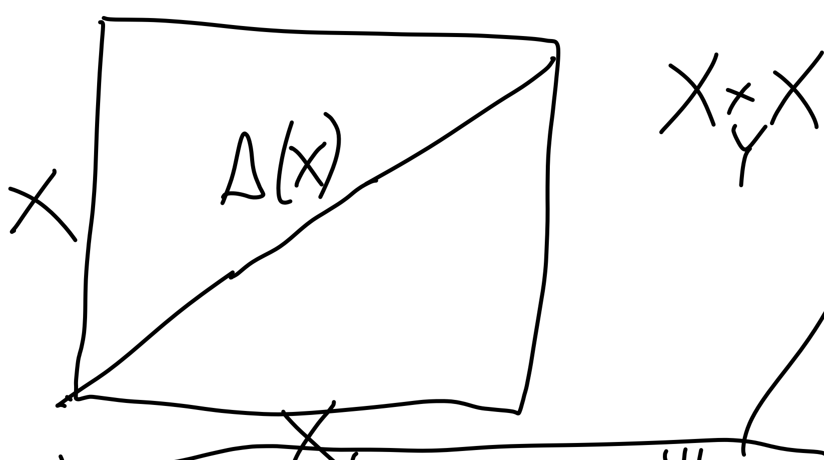
Why does this help with defining $\Omega_{X/Y}$?

Suppose $\pi : X \to Y$ is seperated, so

$$\Delta : X \to X \times_Y X \text{ is a closed embedding.}$$

Then we the relative cotangent sheaf as

$$\Omega_{X/Y} := \mathcal{N}^\vee_{X/X \times_Y X} .$$



intuition

$$N_{X/X\times X} = T_{X\times X}/T_{\Delta(X)}$$

$$= T_X \times T_X / \Delta(T_X)$$

$$\cong T_X .$$

Algebra check: this agrees with earlier description of $\Omega_{A/B}$.