<u>Today</u>: half technical results, half examples

<u>Last time</u>: two exact sequences:

Given $X \xrightarrow{\pi} Y \to Z$:

$$\pi^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

affines: $$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0$$

$$a \otimes db \longmapsto a\,db$$

$$da \longmapsto da$$

If π is a closed embedding:

$$N^\vee_{X/Y} \to \pi^* \Omega_{Y/Z} \to \Omega_{X/Z} \to 0$$

affines: $$I/I^2 \to B/I \otimes_B \Omega_{B/C} \to \Omega_{(B/I)/C} \to 0$$

$$i \longmapsto 1 \otimes di.$$

When are these left-exact?

Happy if X is integral and everything is a vector bundle with ranks adding up.

Why? Algebra result:

<u>Lemma:</u> A is a domain

$$A^m \to A^{m+n} \to A^n \to 0 \quad \text{exact (unknown morphisms)}$$

Then this is exact on the left as well.

<u>Pf:</u> Let M be the kernel on the left. Then M is torsion-free (submodule of A^m), so $M \hookrightarrow M \otimes_A K(A)$ is an injection. But

$$0 \to M \otimes_A K(A) \to K(A)^m \to K(A)^{m+n} \to K(A)^n \to 0$$

is exact, so $M \otimes_A K(A) = 0$, so $M = 0$. $\blacksquare$.

Relative cotangent sequence: exact on left if $X \to Y$, $Y \to Z$ are "smooth morphisms".

We will see a good variant on this later.

Today: conormal exact sequence:

Thm: Suppose $i: X \hookrightarrow Y$ is a closed embedding of smooth k-varieties. Then the conormal sequence is exact on the left:

$$0 \longrightarrow N^{\vee}_{X/Y} \longrightarrow i^* \Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow 0.$$

Pf: By the above discussion/Lemma, it suffices to check that $N^{\vee}_{X/Y}$ is a vector bundle of rank $\dim Y - \dim X$

[This follows from notion of a "regular sequence": can check locally in the stalk at every point that the ideal $I \subset \mathcal{O}_{Y,p}$ cutting out X inside Y can be generated by elts $x_1, \dots, x_r$ s.t. $r = \dim Y - \dim X$ and x_i is not a zero-divisor in $\mathcal{O}_{Y,p}/(x_1, \dots, x_{i-1})$, and then $(I/I^2)/m_p(I/I^2) \cong \bigoplus_{i=1}^{r} k \cdot x_i$.] $\boxtimes$

Cor: Suppose Y is a smooth k-variety and $q \in Y$ is a closed point s.t. k_q is separable over k. Then the conormal exact sequence yields an isomorphism

$$\underbrace{T_{Y,q}^{\vee}}_{\text{cotangent space at } q} \cong \underbrace{\Omega_{Y/k}\big|_q}_{\text{fiber of cotangent sheaf at } q.} = i^* \Omega_{Y/k}$$

Pf: Compute $\Omega_{k_q/k} = 0$, so $\operatorname{Spec} k_q$ is smooth$/k$. Then apply prev. thm.

(Confusing note: If k_q is inseparable over k, still the case that $T_{Y,q}^{\vee}$ and $\Omega_{Y/k}|_q$ will be k_q-vector spaces of the same rank $\dim Y$. But they aren't naturally isomorphic (at least by this map).)

Prop: Suppose K/k is a field extension of trans. degree n that is sep. generated by $t_1, \dots, t_n \in K$, i.e. $K/k(t_1, \dots, t_n)$ is a sep. alg. extension. Then $\Omega_{K/k} \cong \bigoplus_{i=1}^{n} K \cdot dt_i$.

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Cor: (Generic smoothness): Suppose X is an irred. variety of dim n over a perfect field k. Then some dense open $U \subseteq X$ is smooth/k (of dim n).

Pf: $K(X)/k$ can be sep. generated by n elements (hard alg. fact if char $k > 0$), so $\Omega_{K(X)/k} \cong K(X)^{\oplus n}$. But this is the fiber of $\Omega_{X/k}$ at its generic point η, so by semicontinuity $\Omega_{X/k}$ is a rank n vector bundle in some open nhood of η. $\boxed{}$.

Applications of the conormal sequence:

Suppose Z is a smooth subvariety of a smooth variety X.

Then we have

$$0 \to N^\vee_{Z/X} \to \iota^*\Omega_{X/k} \to \Omega_{Z/k} \to 0$$

$$\iota^*\Omega_{X/k} = \Omega_X|_Z, \quad \Omega_{Z/k} = \Omega_Z,$$

(Note: can dualize everything to get

$$0 \to T_Z \to \iota^* T_X \to N_{Z/X} \to 0)$$

T_Z = tangent sheaf, $N_{Z/X}$ = normal sheaf)

Q: If Z is codim 1 in X, then $N^\vee_{Z/X}$ is a line bundle. Do we know which line bundle this is?

A: Yes!

$$N^\vee_{Z/X} \underset{\text{def}}{=\!=} (\tilde{I}/\tilde{I}^2)\Big|_Z \underset{\text{claim}}{=\!=} \tilde{I}\Big|_Z \cong \mathcal{O}_X(-Z)\Big|_Z$$

(claim follows from $(I/I^2) \otimes_A (A/I) \cong I/I^2 \cong I \otimes_A (A/I)$)

So suppose now $D \subset X$ is a smooth divisor (codim 1) in a smooth k-variety X.

Then

$$0 \longrightarrow \underbrace{\mathcal{O}_X(-D)\Big|_D}_{\substack{\text{line bundle} \\ \text{rank } 1}} \longrightarrow \underbrace{\Omega_X\Big|_D}_{\text{rank } \dim X} \longrightarrow \underbrace{\Omega_D}_{\text{rank } \dim X - 1} \longrightarrow 0.$$

Now: want to think about everything in terms of line bundles.

<u>Def</u>: Let $\mathcal{F}$ be a vector bundle of rank r on a scheme X. Then there is a natural line bundle (<u>determinant bundle</u>) on X, denoted $\det \mathcal{F}$, which can be described either as the top exterior power $\Lambda^r \mathcal{F}$ or via transition functions: if $\mathcal{F}$ is given by transition functions $t_{ij} \in GL_r(\mathcal{O}_X(U_i \cap U_j))$, then $\det \mathcal{F}$ is given by $t'_{ij} = \det t_{ij}$.

<u>Lemma</u>: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of vector bundles, then there is a canonical isom of line bundles

$$\det \mathcal{G} \cong \det \mathcal{F} \otimes \det \mathcal{H}.$$

<u>Def</u> (important): The <u>canonical bundle</u> of a smooth k-variety X is the line bundle

$$K_X = \omega_X := \det \Omega_{X/k}.$$

(equal to dualizing sheaf in Serre duality)

Prop ("adjunction formula"): If D is a smooth divisor in a smooth variety X, then

$$K_X|_D \cong K_D \otimes (\mathcal{O}_X(-D)|_D).$$

Examples:

1) Suppose $D \cong \mathbb{P}^1_k$ is a smooth conic in $X = \mathbb{P}^2_k$.
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Then $K_{\mathbb{P}^2}|_D \cong K_D \otimes (\mathcal{O}_{\mathbb{P}^2}(\underset{1}{-2})|_D)$

We know $K_{\mathbb{P}^1_k} = \Omega_{\mathbb{P}^1_k} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. Also, we know the map $|_D : \underset{\mathbb{Z}}{\operatorname{Pic}(\mathbb{P}^2_k)} \to \operatorname{Pic}(D) \cong \underset{\mathbb{Z}}{\operatorname{Pic}(\mathbb{P}^1_k)}$

is mult by $\underset{1}{2}$, since $i^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_{\mathbb{P}^1}(\underset{1}{2})$.

So if $K_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(d)$, we have

$$\mathcal{O}_{\mathbb{P}^1}(\underset{d}{2d}) \cong \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(\underset{1}{-4}),$$

so $d = -3$ and we conclude that

$K_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$.

2) Can use $D =$ hyperplane inside $\mathbb{P}^n$ to prove by induction that $K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ for all n. (We just saw $n=1 \Rightarrow n=2$)

3) Suppose D is a smooth plane curve of deg d.

Then

$$K_D \cong \Big(\overbrace{K_{\mathbb{P}^2}}^{\mathcal{O}(-3)}\big|_D\Big) \otimes \Big(\overbrace{\mathcal{O}_{\mathbb{P}^2}(D)}^{\mathcal{O}_{\mathbb{P}^2}(d)}\big|_D\Big)$$

$$\cong \mathcal{O}_{\mathbb{P}^2}(d-3)\big|_D \cong \left(\mathcal{O}_{\mathbb{P}^2}(1)\big|_D\right)^{\otimes(d-3)}$$

is some line bundle of degree $d(d-3)$.

This agrees with $g(D) = \binom{d-1}{2}$, since we expect

$$\deg K_D = \deg \omega_D = 2g(D) - 2.$$

(Also get this way that $K_C \cong \mathcal{O}_C$ for any smooth cubic plane curve.)