<u>Last time</u>: A loc. finite type morphism $\pi: X \to Y$ is <u>unramified</u> if $\Omega_{X/Y} = 0$

- "no vertical cotangent vectors"
- fiber over a geometric point $\operatorname{Spec} K \to Y$, $K = \bar{K}$ is a disjoint union of copies of $\operatorname{Spec} K$.
- most of the way towards being the analogue of a covering space map.

<u>Defs</u>: Let $\pi: X \to Y$ be a dominant morphism of integral schemes (i.e. $\pi(\eta_X) = \eta_Y$), so we have a field extension $K(X)/K(Y)$.

We say π is <u>generically finite</u> if $K(X)/K(Y)$ is finite,

π is <u>generically separable</u> if $K(X)/K(Y)$ is separable,

π is <u>separable</u> if π is finite and generically separable.

($\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ is gen. finite but not finite).

Prop: If π is gen. separable, it is unramified on some dense open.

Pf: $\Omega_{X/Y}$ is 0 at η_X, so 0 on some dense open. $\blacksquare$

So in this case the ramification locus $\mathrm{Supp}\,\Omega_{X/Y}$ is codim ≥ 1. We want something stronger.

Prop: Suppose $\pi : X \to Y$ is a gen. separable morphism of irred smooth k-varieties. Then the relative cotangent sequence is exact on the left;

$$0 \to \underbrace{\pi^*\Omega_{Y/k} \to \Omega_{X/k}}_{\text{vector bundles of the same rank}} \to \underbrace{\Omega_{X/Y}}_{\text{supported in codim} \geq 1} \to 0 .$$

Pf: We use the same technique as with conormal sequence: the map of sheaves $\pi^*\Omega_{Y/k} \to \Omega_{X/k}$ has torsion-free kernel (since contained in a vector bundle) and the kernel is 0 at η_X (looks like $K(X)^n \to K(X)^n \to 0 \to 0$), so the kernel is the 0 sheaf. $\blacksquare$

Cor: Under these assumptions, the ramification locus $\operatorname{Supp} \Omega_{X/Y}$ is pure codim 1.

Pf: Over an affine open $U = \operatorname{Spec} A \subseteq X$ trivializing both $\pi^* \Omega_{Y/k}$ and $\Omega_{X/k}$, we have that $\Omega_{X/Y}\big|_U$ corresponds to the A-module that is cokernel of some linear map $A^{\oplus n} \longrightarrow A^{\oplus n}$.

This will be supported precisely where the determinant (of an $n \times n$ matrix/A) vanishes, i.e. on a set cut out by a single equation. $\blacksquare$

Def: The __ramification divisor__ is the effective Cartier divisor cut out in this way. The corresponding Weil divisor is denoted $R_\pi \in \operatorname{Weil} X$.

Example: $\pi : \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$. What is R_π?

$x \longmapsto x^n$

$$\operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k[y]$$

$$x^n \longleftarrow y$$

Relative cotangent sequence:

$$0 \longrightarrow \Omega_{k[y]/k} \otimes_{k[y]} k[x] \longrightarrow \Omega_{k[x]/k} \longrightarrow \Omega_{k[x]/k[y]} \to 0$$

$$\begin{array}{ccc} \| & & \| \\ k[x]\cdot dy & & k[x]\cdot dx \end{array}$$

$$dy \longmapsto d(x^n) = nx^{n-1}dx$$

So $R_\pi = V(nx^{n-1}) = (n-1)\cdot[0] \in \operatorname{Weil} \mathbb{A}^1_{\mathbb{C}}$.

<u>Can check</u>: If X and Y are smooth curves and

$\pi: X \to Y$ is a dominant morphism, then
$p \mapsto q$
closed point

the multiplicity of R_π at p is "usually"

$v_p(\pi^* t_q) - 1$, where $t_q \in \mathcal{O}_{Y,q}$ is a uniformizer

and v_p is the valuation on $\mathcal{O}_{X,p}$

Why only "usually"? might have $\operatorname{char} k \mid v_p(\pi^* t_q)$.

("wild ramification").

(Note: if $\operatorname{char} k = 0$, multiplicity of p in $R_\pi \leq \deg \pi - 1$)

Thm (Riemann-Hurwitz): Suppose $\pi: X \to Y$ is a separable morphism of degree d of geom. integral proj. smooth curves/k. Then

$$\underbrace{2g_X - 2}_{} = \underbrace{d}_{\text{deg of morphism}}(2g_Y - 2) + \underbrace{\deg(R_\pi)}_{\text{degree of ramif. divisor}}$$

genera of X, Y

Pf: Euler characteristic is additive in short exact sequences (easy, proved)

+ $\deg K_X = \deg \Omega_{X/k} = 2g_X - 2$ (follows from R-R, still yet to prove)

+ $\deg(\pi^* \mathcal{L}) = d \cdot \deg(\mathcal{L})$ (exercise, not hard):

$$0 \to \pi^* K_Y \to K_X \to \Omega_{X/Y} \to 0,$$

so $\deg K_X - \deg \pi^* K_Y = \chi(X, \underbrace{\Omega_{X/Y}}_{\text{supported in dim } 0})$
$= h^0(X, \Omega_{X/Y})$. ∎

Consequences of Riemann-Hurwitz (X, Y smooth proj integral curves)

1) If $g_X < g_Y$, every morphism $X \to Y$ is constant, since $2g_X - 2 - d(2g_Y - 2) < 0$, but $R_\pi \geq 0$, so this is a contradiction if R-H applies. (if char $k > 0$, needs a bit of extra work studying "purely inseparable morphisms".)

2) If $k = \bar{k}$ and $Y = \mathbb{P}^1_k$ and $R_\pi = 0$ (π is unramified), then $X \to Y$ is the identity (since R-H says $2g_X - 2 = d(-2)$, so $g_X = 0$, $d = 1$).

Interpretation: "$\mathbb{P}^1_k$ is simply connected".

3) The same result holds if $k = \mathbb{C}$ and $\pi_* R_\pi = m \cdot [\infty] \in \mathbb{P}^1_\mathbb{C}$, i.e. $\pi : X \to \mathbb{P}^1_\mathbb{C}$ is branched only over ∞.

Why? char $k=0 \Rightarrow m \leq d-1$, and then

R-H: $2g_X - 2 \leq d(-2) + (d-1)$

$\Longrightarrow d=1, g_X = 0$.

Interpretation: "$\mathbb{A}^1_{\mathbb{C}}$ is simply connected", since an unramified cover $X \to \mathbb{A}^1_{\mathbb{C}}$ can be completed to get a map branched only over ∞.

Can keep on going, and compute

"$\pi_1(\mathbb{A}^1_k)$" for char $k=p$.

or "$\pi_1(\mathbb{A}^1_{\mathbb{C}} - \{0\})$"

$\Bigg\}$ nontrivial, i.e. $\exists$ nontrivial unramified connected covers.

4) return to hyperelliptic and trigonal curves:

We previously computed that a double cover $C \to \mathbb{P}^1_k$ is branched over $2g_C+2$ points.

This is now immediate: R–H gives
$$\deg R_\pi = 2g-2-2(2\cdot 0-2) = 2g+2.$$

Similarly, if $\pi: C \to \mathbb{P}^1_k$ is deg 3, then $\deg R_\pi = 2g+4$.

So if we knew that any choice of $2g+4$ distinct points in $\mathbb{P}^1_k$ had a positive finite number of branched triple covers branched over those points, we could conclude that

$$\dim \{\text{trigonal curves of genus } g\}/\text{iso}$$
$$= 2g+4 - \dim \operatorname{Aut}(\mathbb{P}^1) = 2g+1,$$

for $g \geq 4$. ["how many parameters are there for a branched cover"

5) "Algebraic genus = topological genus":

you can state and prove a topological version of Riemann-Hurwitz (using top. Euler characteristic), and then the two copies of R-H along with the existence of a dominant map $C \to \mathbb{P}^1_{\mathbb{C}}$ for any smooth proj curve $C/\mathbb{C}$ will yield alg. genus = top. genus.