<u>Today</u>: beginning of proof of Riemann-Roch;

Setup: C = smooth proj integral curve/k, $k=\bar{k}$

$K = K(C)$.

L = line bundle on C

$\cong \mathcal{O}_C(D)$ for some $D \in \mathrm{Weil}(C)$.

<u>R-R</u>: $h^1(C,L) = h^0(C, \Omega_{C/k} \otimes L^\vee)$

(reminder: $h^0(C,L) - h^1(C,L) = \chi(C,L) = \chi(C,\mathcal{O}_C) + \deg L$)

puts this in the more commonly used form

$$h^0(C,L) - h^0(C, \Omega_{C/k} \otimes L^\vee) = d - g + 1$$

We'll actually prove the "Serre duality for curves and line bundles" version:

$$H^1(C,L) \underset{\text{canon.}}{\cong} H^0(C, \Omega_{C/k} \otimes L^\vee)^\vee$$

We want to define a perfect pairing of k-vector spaces

$$H^1(C,L) \times H^0(C, \Omega_C \otimes L^\vee) \longrightarrow k.$$

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3 ingredients in the proof:

1) explicit description of $H^1(C,L)$ in the language of repartitions ($\sim$ adeles)

2) definition of residues of differentials $\left(\mathrm{Res}_0 \frac{dz}{z} = 1\right)$

3) The Residue Thm $\left(\sum_{p \in C} \mathrm{Res}_p \, \alpha = 0\right)$

Ingredient 1 is purely algebraic, while 2 and 3 are easy complex analysis statements that need to be transported to algebraic geometry.

(As before: take $L = \mathcal{O}_C(D)$ for $D =$ finite linear comb of closed points in C.)

How can we compute $H^1(C, \mathcal{O}_C(D))$?

Recall:

$$H^0(U, \mathcal{O}_C(D)) = \{ f \in K \mid \operatorname{div}(f) + D \geq 0 \text{ on } U \} \subset K.$$

In other words, there is a morphism of qcoh sheaves (on C)

$\mathcal{O}_C(D) \longrightarrow \underline{K}$, where $\underline{K}$ is the constant sheaf $\underline{K}(U) = \begin{cases} K & \text{if } U \neq \emptyset \\ 0 & \text{if } U = \emptyset \end{cases}$.

This is injective on every affine open, so there is some s.e.s. of qcoh sheaves on C:

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \underline{K} \longrightarrow \mathcal{F} \longrightarrow 0$$

Long exact sequence:

$$0 \to \underset{?}{\underset{\|}{H^0(C,\mathcal{O}_C(D))}} \to \overset{K}{\overset{\|}{H^0(C,\underline{K})}} \to H^0(C,\mathcal{F})$$

$$\to \underbrace{H^1(C,\mathcal{O}_C(D))}_{\text{we want this}} \to \underset{0}{\underset{\|}{H^1(C,\underline{K})}} \to \cdots$$

So $H^1(C,\mathcal{O}_C(D)) = \operatorname{coker}\big(K \to H^0(C,\mathcal{F})\big)$.

So we now just need to understand
$H^0(C,\mathcal{F})$ $\quad (\mathcal{F} = \operatorname{coker}(\mathcal{O}_C(D) \to \underline{K}))$.

How to understand $\mathcal{F} = \operatorname{coker}(\mathcal{O}_C(D) \to \underline{K})$?

The stalks of $\mathcal{F}$ can be computed:

If p is closed:

$$\mathcal{F}_p = \underline{K}_p / \mathcal{O}_C(D)_p$$

$$= K / \{ f \in K \mid \operatorname{div}(f) + D \geq 0 \text{ "at } p\text{"}, \text{ i.e. } v_p(f) + \operatorname{coeff}_p D \geq 0 \}$$

$$\boxed{\text{If } p = \eta \text{ is the generic point: } \mathcal{F}_\eta = 0.}$$

Prop: $\mathcal{F} \cong \bigoplus_{\substack{p \in C \\ \text{closed}}} \underbrace{(\iota_{p*} \mathcal{F}_p)}_{\text{skyscraper sheaf}}$ for $\iota_p : \operatorname{Spec} k \to C$, $* \mapsto p$

Cor: $H^0(C, \mathcal{F}) \cong \bigoplus_{\substack{p \in C \\ \text{closed}}} \mathcal{F}_p$.

Pf of prop:

Suppose $s \in \tilde{\mathcal{F}}_p$, p closed.

Then s can be realized by some $\tilde{s} \in \tilde{\mathcal{F}}(U)$, $U \ni p$.

But since $(\tilde{s})_\eta = 0$, $\tilde{s}|_{U'} = 0$ for some $U' \subset U$.

But then consider $\tilde{s}|_{U''}$ for $U'' = U' \cup \{p\}$.

This satisfies $\left(\tilde{S}|_{U''}\right)\Big|_{(U'' - \{p\})} = 0$, so can glue

with the 0 section on $C \setminus \{p\}$ to get a section

$\bar{s} \in \tilde{\mathcal{F}}(C)$ with $\bar{s}|_{C \setminus \{p\}} = 0$ but $(\bar{s})_p = s$.

So we've defined a map

$\bigoplus_{\substack{p \in C \\ \text{closed}}} (\iota_{p*} \tilde{\mathcal{F}}_p) \longrightarrow \tilde{\mathcal{F}}$, and it's an isomorphism since any section of $\tilde{\mathcal{F}}$ vanishes on the complement of finitely many points. ☒

<u>Def</u>: Let $T_i \subseteq S_i$ be pairs of sets indexed by some I $(i \in I)$. The the <u>restricted product</u> is

$$\underbrace{\prod_{i\in I}{}' S_i}_{\text{choice of } T_i\text{'s is implicit in notation}} = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} S_i \;\middle|\; \begin{array}{l} x_i \in T_i \text{ for} \\ \text{all but finitely} \\ \text{many } i \end{array} \right\}.$$

(Example: if S_i's are abelian groups and $T_i = 0$, $\prod' = \bigoplus$.)

<u>Def</u>: The ring of <u>repartitions</u> of C is the restricted product

$$R := \prod_{\substack{p\in C \\ \text{closed}}}{}' K = \left\{ (f_p)_{\substack{p\in C \\ \text{closed}}} \;\middle|\; \begin{array}{l} f_p \in \mathcal{O}_{C,p} \subset K \text{ for} \\ \text{all but finitely many } p \end{array} \right\},$$

i.e. f_p has a pole at p for only finitely many p.

<u>Notation</u>: for $D \in \mathrm{Weil}(C)$,

$$R(D) := \left\{ (f_p) \in R \;\middle|\; v_p(f_p) + \mathrm{coeff}_p(D) \geq 0 \text{ for all } p \in C_{\text{closed}} \right\}$$

<u>Cor</u>: $H^0(C, \mathcal{F}) \cong R/R(D)$.

<u>Pf</u>: $$\frac{R}{R(D)} = \frac{\prod'_{p \in C} (K \supseteq \mathcal{O}_{C,p})}{\prod_{p \in C} \underbrace{\{ f \in K \mid v_p(f) + \mathrm{coeff}_p(D) \geq 0 \}}_{A_p}}$$

$$= \prod'_{p \in C} \Big(K/A_p \supset \underbrace{\mathcal{O}_{C,p}/\mathcal{O}_{C,p} \cap A_p}_{= 0 \text{ for all but finitely } p \in C} \Big)$$

$$= \bigoplus_{p \in C} K/A_p.$$

Recall what we wanted was

$$H^1(C, \mathcal{O}_C(D)) \cong \operatorname{coker}\left(K \longrightarrow H^0(C, \mathcal{F})\right)$$

This map is easy to describe —

it is induced from the diagonal inclusion

$$K \hookrightarrow R.$$

Conclusion:

<u>Prop</u>! $H^1(C, \mathcal{O}_C(D)) \cong R / (\underbrace{R(D)}_{\text{k-v.s. spanned by these.}} + K)$

<u>Plan</u>: Can interpret $H^0(C, \Omega_C(-D))$ as a subspace of $M = \{\text{rational sections of } \Omega_C\}$ ("meromorphic differentials").

Consider a k-bilinear pairing

$$\langle \cdot, \cdot \rangle : \underbrace{R \times M}_{k\text{-v.s.}} \longrightarrow k \text{ defined by}$$

$$\langle (r_p), \alpha \rangle := \sum_p \operatorname{Res}_p (\underbrace{r_p \cdot \alpha}_{\substack{\text{has a pole at } p \text{ for} \\ \text{only finitely many } p.}}).$$

Def: Suppose $\alpha \in M$ is a meromorphic differential and p is a closed point of C. The residue of α at p is defined as follows:

Choose a uniformizer t at p $(v_p(t)=1)$.

Then

$$\underbrace{\alpha = \left(\sum_{i \geq -N} c_i t^i\right) dt}$$ for some $N>0$ and $c_i \in k$ (since $k_p = k$).

Take $\mathrm{Res}_p\, \alpha := c_{-1}$ (the coeff of $\frac{dt}{t}$).

Prop: This def does not depend on the choice of t.

Pf: Suppose $t = \sum_{j \geq 1} b_j u^j$ for $b_j \in k$. Then the invariance of the def of $\mathrm{Res}_p\, \alpha$ is equiv to:

$$c_{-1} = \left[\left(\sum_{i \geq -N} c_i \left(\sum_{j \geq 1} b_j u^j \right)^i \right) \left(\sum_{j \geq 1} j b_j u^{j-1} \right) \right]_{u^{-1}}.$$

Both sides are in $\mathbb{Z}[\{c_i\}, \{b_i\}, b_1^{-1}]$, and identity holds in $\mathbb{C}$ by complex analysis, so it holds in any field k. ∎