Recall: proving Riemann-Roch ($H^1(C,\mathcal{O}_C(D)) \cong H^0(C,\Omega_C(-D))^\vee$)

Setup/Notation: $C/k$, $L = \mathcal{O}_C(D)$, $K = K(C)$

$$R = \left\{ (f_p)_{\substack{p\in C \\ \text{closed}}} \in \prod_{p \in C} K \;\middle|\; f_p \in \mathcal{O}_{C,p} \text{ for all but finitely many } p \right\}$$

$$R(D) = \{ (f_p)_{p\in C} \in R \mid v_p(f_p) + D \geq 0 \text{ at } p \}$$

$$H^1(C,\mathcal{O}_C(D)) \cong R/(R(D) + K)$$
($K$ embedded by diagonal)

$$M = (\Omega_C)_\eta = \{\text{meromorphic differentials on } C\}$$

$$H^0(C,\Omega_C(-D)) \cong M(-D) := \{\alpha \in M \mid \operatorname{div}(M) - D \geq 0\}.$$

Idea: residue pairing on $R \times M$ should induce one on $R/(R(D)+K) \times M(-D)$.

This requires checking that

$$\underset{\sum_p \operatorname{Res}_p(r_p \cdot \alpha)}{\underset{\shortparallel}{\langle r, \alpha \rangle}} = 0 \text{ for } r \in R(D), \alpha \in M(-D)$$
$$\text{or } r \in K, \alpha \in M(-D).$$

Easy: $\langle r, \alpha \rangle = 0$ for $r \in R(D)$, $\alpha \in M(-D)$, since then $r_p \cdot \alpha$ never has a pole at $p$.

Hard:

Residue Thm: Suppose $\alpha \in M$. Then

$$\sum_{p \in C} \operatorname{Res}_p \alpha = 0.$$

Pf: delayed to Tuesday.

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So assuming Residue Thm, we've defined a $k$-bilinear pairing

$$H^1(C, \mathcal{O}_C(D)) \times H^0(C, \Omega_C(-D)) \longrightarrow k.$$

(can check: doesn't depend on choice of $D$, just on $L = \mathcal{O}_C(D)$.).

We wish to show that the induced map of $k$-vector spaces

$$\begin{array}{ccc} H^0(C, \Omega_C(-D)) & \longrightarrow & H^1(C, \mathcal{O}_C(D))^\vee \\ \cong & & \\ M \supseteq M(-D) & \longrightarrow & \left(R/(R(D)+K)\right)^\vee \end{array}$$

is an isomorphism.

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Injectivity is easy: given $\alpha \in M(-D)$, $\alpha \neq 0$, we can find $r = (r_p) \in R$ with which it pairs nontrivially by taking $r_p = \begin{cases} t^n & \text{for } p = p_0 \\ 0 & \text{else} \end{cases}$,

where $t$ is a uniformizer for $\mathcal{O}_{C,p_0}$, $n = v_{p_0}(\alpha) - 1$.

(So $r_{p_0} \cdot \alpha$ has a simple pole at $p_0$, so nonzero residue there.)

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(Remark: injectivity would be enough by a symmetry argument if we knew $\deg \Omega_C = 2g-2$, but we don't (yet).)

Surjectivity is harder.

<u>Notation:</u> $J(-D) := \left(R/(R(D)+K)\right)^\vee$,

so we want to show our maps

$M(-D) \to J(-D)$ are surjective.

Each $J(-D)$ is naturally a $k$-vector subspace of $R^\vee = \mathrm{Hom}_k(R,k)$, namely those functionals vanishing on both $R(D)$ and $K$.

Since $R(D) \subseteq R(D')$ for $D \leq D'$, we have

$J(-D') \subseteq J(-D)$ for $D \leq D'$ ($\Leftrightarrow -D' \leq -D$)

<u>Def</u>: $J := \bigcup_{D \in \mathrm{Weil}\, C} J(-D)$ $\left(\subseteq R^\vee\right)$, a $k$-vector space by the above.

So the morphisms $M(-D) \to J(-D)$ glue together on both sides to a single morphism

$\Theta : M \to J$.

(Note: $\Theta(\alpha) = \left( (r_p)_{p\in C} \longmapsto \sum_{p\in C} \operatorname{Res}_p(r_p \alpha) \right)$)

Any $\alpha \in M$ is in some $M(-D)$, and then this $\Theta(\alpha)$ will vanish on both $R(D)$ and $K$, and thus belong to $J(-D) \subseteq J$.

<u>Note</u>: $\underset{\substack{\shortparallel \\ \text{rational sections}}}{M} \neq M(0) = H^0(C, \Omega_C)$

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<u>Lemma</u>: $\Theta^{-1}(J(-D)) = M(-D)$.

<u>Cor</u>: If $\Theta$ is surjective, then each $\Theta|_{M(-D)} : M(-D) \to J(-D)$ is also surjective.

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<u>Pf of Lemma</u>: If $\alpha \in M \setminus M(-D)$, then one of the $(r_p) = \begin{cases} t^n & \text{if } p = p_0 \\ 0 & \text{else} \end{cases}$ $(n = v_{p_0}(\alpha) - 1)$ will be in $R(D)$. So then $\Theta(\alpha)((r_p)) \neq 0$, i.e. $\Theta(\alpha)$ doesn't vanish on $R(D)$, so $\Theta(\alpha) \notin J(-D)$. $\blacksquare$

It remains to show that

$\Theta : M \to J$ is surjective.

- This is actually a morphism of $K$-vector spaces,

where $K$ acts on $J \subset R^\vee = \mathrm{Hom}_k(R, k)$ by

$$\underbrace{(f \cdot l)}_{\substack{\text{defining} \\ K\text{-v.s. structure} \\ \text{on } R^\vee}}(r) := l(\underbrace{f \cdot r}_{K\text{-v.s structure on } R})$$

So we just need to prove (since $\Theta$ is clearly not the $0$ morphism)

<u>Prop</u>: $\dim_K J \leq 1$.

(After this, we will have proven R-R assuming still the Residue Thm.)

Pf of Prop:

Suppose otherwise, so $l_1, l_2 \in J$ are lin. independent over $K$.

Then we can choose $D$ s.t. $l_1, l_2 \in J(-D)$.

Let $\Delta \geq 0$ be some effective Weil divisor.

Suppose $f_1, f_2 \in K$ correspond to global sections of $\mathcal{O}_C(\Delta)$, i.e. $\operatorname{div}(f_i) + \Delta \geq 0$.

Then can check that $f_1 l_1, f_2 l_2 \in J(\Delta - D)$.

So then we have an injection (by linear dependence)

$$\begin{aligned} H^0(C, \Delta) \oplus H^0(C, \Delta) &\hookrightarrow J(\Delta - D) \\ (f_1, f_2) &\longmapsto f_1 l_1 + f_2 l_2. \end{aligned}$$

So $2h^0(C, \Delta) \leq h^1(C, D - \Delta)$ for all effective $\Delta \geq 0$ in Weil $C$.

We know

$$h^0(C,L) - h^1(C,L) = \deg L - g + 1$$

for every line bundle $L$ on $C$,

so this becomes (letting $N = \deg \Delta$, $d = \deg D$)

$$2(N-g+1) \le 2h^0(C,\Delta) \le h^1(C, D-\Delta) \le -(d-N-g+1).$$

But this is false for $N \gg 0$. $\boxtimes$.

Remarks on replacing $C$ by $X$ of dim $n$.

We expect isomorphisms between

$$J(-D) := \underbrace{H^n(X, \mathcal{O}_X(D))^\vee}_{\text{expect also glue into some } J,} \text{ and } \underbrace{H^0(X, K_X(-D))}_{\text{still glue together to } M = \text{meromorphic } n\text{-forms, i.e. } (K_X)_\eta, \text{ a 1-dim } K\text{-vector space}}$$

If $D \le D'$, the short exact sequence

$$0 \to \mathcal{O}_X(-D') \to \mathcal{O}_X(-D) \to \underbrace{\begin{pmatrix}\text{sheaf supported} \\ \text{on } D'-D\end{pmatrix}}_{=\mathcal{F}} \to 0$$

gives a long exact sequence, and

$$H^n(X, \mathcal{F}) = 0 \text{ by dim. vanishing,}$$

so get surjections $H^n(X, \mathcal{O}_X(-D')) \to H^n(X, \mathcal{O}_X(-D))$.

Hence get injections

$$J(D) \to J(D') \quad \text{whenever } D \leq D'.$$

Define $J$ to be the colimit of these injections.

Can then prove (with a little work) that

$J$ is a $K$-vector space, and then

again prove $\dim_K J = 1$.

Argument is analogous except that we require that $X$ is projective and replace $\mathcal{O}_C(\Delta)$ with $\mathcal{O}_X(N)$ (pulled back from some $\mathbb{P}^{N'}$).

Then need bounds both on $h^0(X, \mathcal{O}_X(N))$ ] hilb poly of $X$

and on $h^n(X, L \otimes \mathcal{O}_X(-N))$ ] $\widetilde{\text{hilb}}$ poly

Again, all that the prop needs is "asymptotic" info on $h^i$'s.