Final scheduled office hours: today at 3pm
– but still feel free to e-mail me.

psets 6/7 – should finish grading later this week

Last time: finished proving Riemann-Roch, with one remaining loose end:

Residue Thm: Let C be a smooth proj curve over $k = \bar{k}$. Let α be a rational section of $\Omega_{C/k}$. Then

$$\sum_{\substack{p \in C \\ \text{closed}}} \mathrm{Res}_p\, \alpha = 0.$$

Shape of the proof:

1) check for $C = \mathbb{P}^1_k$

2) describe a compatibility between residues on C and C' if $\pi : C \to C'$ is finite separable.

This compatibility would let us lift the Residue thm for $\mathbb{P}^1_k$ to any C. (via some $\underbrace{C \to \mathbb{P}^1_k}_{k(t) \hookrightarrow K(C)}$).

3) reduction from $k(t) \hookrightarrow K(C)$ to $k((t)) \hookrightarrow k((u))$

$$\sum_{i \geq -N} c_i t^i \in k((t)) \qquad t \longmapsto u^r + \text{higher order in } u.$$

4) explicit computation for the $k((t)) \hookrightarrow k((u))$ compatibility.

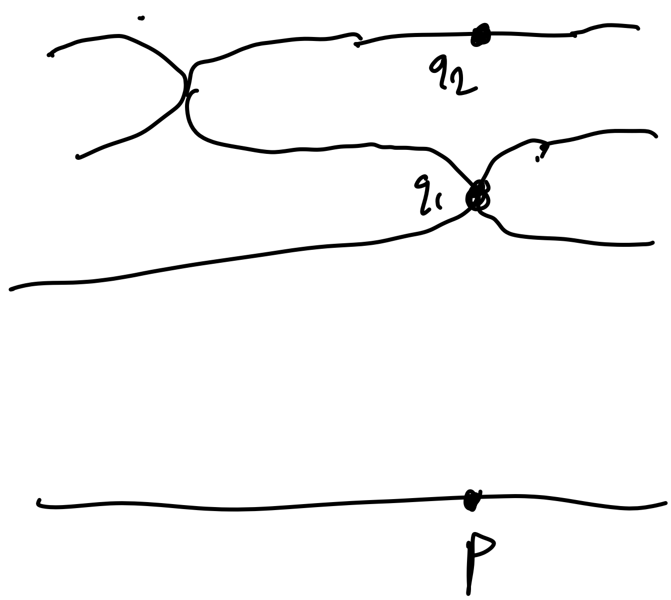
<u>Lemma</u>: Residue Thm holds for $C = \mathbb{P}^1$.

<u>Pf</u>: direct computation; partial fractions means it suffices to check for

$$\alpha = t^n \, dt$$

and $$\alpha = \frac{dt}{(t-a)^n}.$$

$\blacksquare$



$$\begin{array}{c} C \\ \Big\downarrow \pi \text{ finite separable} \\ \mathbb{P}^1 \end{array}$$

given $\alpha \in \Omega_{C,\eta}$

want to construct

$\alpha' \in \Omega_{\mathbb{P}^1,\eta}$

such that $$\operatorname{Res}_p \alpha' = \sum_{q \in \pi^{-1}(p)} \operatorname{Res}_q \alpha$$

<u>Idea</u>: $\pi: C \to \mathbb{P}^1$ corresponds to an inclusion
$$k(t) \hookrightarrow K = K(C). \ \left(\begin{smallmatrix}\text{finite}\\ \text{separable}\end{smallmatrix}\right)$$

We want some sort of map from a
1-dim K-vector space to a
1-dim $k(t)$-vector space.

Want to adjust the trace map $\mathrm{Tr}_{K/k(t)}: K \to k(t)$.
$$f \longmapsto \mathrm{Tr}_{k(t)}(*f : K \to K)$$
(= sum of Galois orbit of f if $K/k(t)$ is Galois).

Need to choose generators for $\Omega_{K/k}$, $\Omega_{k(t)/k}$:

can just use dt for both.

($\neq 0$ in $\Omega_{K/k}$ by separability of $K/k(t)$)

Def: Let $\pi: C \to C'$ be a finite separable morphism of integral smooth proj. curves/k

Let $K = K(C)$, $K' = K(C')$, so have $K' \hookrightarrow K$.

Let $M = \Omega_{C/k,\eta} = \Omega_{K/k}$

$M' = \Omega_{C'/k,\eta'} = \Omega_{K'/k}$.

Then the <u>trace map</u> is the K'-linear map $\mathrm{Tr}: M \longrightarrow M'$ determined by

$$\mathrm{Tr}(a \cdot \pi^* \alpha) = (\mathrm{Tr}_{K/K'}\, a) \cdot \alpha.$$

for any $a \in K$, $\alpha \in M'$.

(for $K' = k(t)$, this says $\mathrm{Tr}(a\, dt) = \mathrm{Tr}(a)\, dt$.

So we want to prove:

Prop: $\displaystyle\sum_{q \in \pi^{-1}(p)} \mathrm{Res}_q \alpha = \mathrm{Res}_p(\mathrm{Tr}(\alpha))$ for any $\alpha \in M$ and $p \in C' (= \mathbb{P}^1)$.

Next step: base change from $K' = k(t)$ to its completion wrt the valuation v_p.

If we take p to be "0", i.e. cut out by t as a closed subscheme of $\mathbb{P}^1$, then this completion is precisely $k((t))$.

So the idea is to take our compatibility of Res and Tr for $K/k(t)$ and apply $\underset{k(t)}{\otimes} k((t))$.

Then $K \underset{k(t)}{\otimes} k((t)) \cong \prod_{q \in \pi^{-1}(p)} K_q$ and

moreover each $K_q \cong k((u))$, where the inclusions

$$k((t)) \hookrightarrow k((u))$$
$$t \longmapsto u^r + \sum_{i>r} c_i u^i.$$

Then the trace formula splits over this product as a sum of traces for these extensions $K_q/k((t))$.

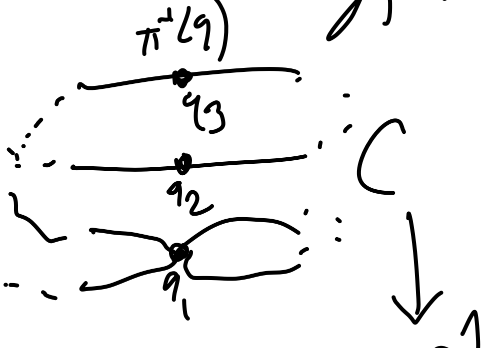
This reduces the prev. prop to needing to prove:

<u>Lemma</u>: Suppose we have a separable extension $k((u))/k((t))$ with $t = u^r + \sum_{i>r} c_i u^i$. Suppose that $f \in k((u))$. Then

$$\left[f \cdot dt\right]_{\frac{du}{u}} = \left[\mathrm{Tr}(f) \cdot dt\right]_{\frac{dt}{t}}$$

take coeffs

Geometrically, the point is that we started with



Then this $\otimes_{k(t)} k((t))$ procedure actually does disconnect the different sheets of π over p

$\pi^{-1}(p) = \{q\}$.

Pf of lemma: $t = u^r + \sum_{i>r} c_i u^i$.

As a $k((t))$-vector space, $k((u))$ has basis $1, u, u^2, \ldots, u^{r-1}$, and mult. by u can be computed in this basis via addition and mult by the c_i only.

So if we write $f = \sum_{i \geq -N} b_i u^i$, both sides of the identity we want to prove are integer coefficient polynomials in the b_i and c_i.

So it suffices to prove the lemma if char $k = 0$.

If char $k = 0$, we can change variables from u to
$$v = \left(u^r + \sum_{i>r} c_i u^i\right)^{1/r}$$
$$= u + \sum_{i>1} c_i' u^i,$$
so our field extension is just $k((t^{1/r}))/k((t))$.

Then easy computation. $\blacksquare$

So given any C and $\alpha \in M$, find a finite separable $\pi : C \to \mathbb{P}^1_k$, and then see

$$\sum_{q \in C} \operatorname{Res}_q \alpha = \sum_{p \in \mathbb{P}^1} \sum_{q \in \pi^{-1}(p)} \operatorname{Res}_q \alpha$$

$$= \sum_{p \in \mathbb{P}^1} \operatorname{Res}_p (\operatorname{Tr}(\alpha))$$

$$= 0. \qquad \blacksquare.$$

We've now proved Riemann–Roch. The proof was in some sense very explicit – let's discuss this.

Setup: C = genus g curve

$p_1, \dots, p_d \in C$ distinct closed points, $d \geq 2g-1$.

$D = [p_1] + \dots + [p_d] \in \mathrm{Weil}\, C$.

Q: What does $H^0(C, \mathcal{O}_C(D))$ look like?

$$\| $$

$$\left\{ \begin{array}{l} \text{rational functions } f \in K(C) \\ \text{that are allowed simple poles} \\ \text{at } p_1, \dots, p_d \end{array} \right\}.$$

R–R: says looks like a k-vector space of dimension $d-g+1$.

(since $h^0(C, \underbrace{\Omega_C(-D)}_{\text{neg degree}}) = 0$).

Given such a rational function f ($\text{so } \operatorname{div}(f)+D \geq 0$), we can take its principal parts at the points $p_1, \ldots, p_d$.

(If t_i is a uniformizer at $p_i \in C$, then we can write $f = \underbrace{\frac{c_i}{t_i}}_{\text{principal part} \in K/\mathcal{O}_{C,p_i}} + \text{something defined at } p_i$).

Let $V := H^0(C, \mathcal{O}_C(D)) / \underbrace{H^0(C, \mathcal{O}_C)}_{\text{constant functions}}$.

Then taking principal parts defines an injection

$$k^{d-g} \cong V \hookrightarrow \left\{ \begin{matrix} \text{principal parts} \\ \text{at } p_1, \ldots, p_d \end{matrix} \right\} \cong k^d.$$

So R-R says that there are precisely g linear conditions describing which sets of principal parts can occur in a rat. function.

Claim: These g linear conditions are given by the residue pairing with g generators for $H^0(C, \Omega_C) \cong k^g$,

i.e. $$\sum_{i=1}^{d} \mathrm{Res}_{p_i}\left(\left(\text{principal part at } p_i \text{ of } f\right) \cdot \alpha\right) = 0$$

for $\alpha \in H^0(C, \Omega_C)$.

So a rat. function with principal parts $\frac{c_i}{t_i}$ at p_i exists

$$\iff \sum_{i=1}^{d} c_i \cdot \mathrm{Res}_{p_i}\left(\frac{1}{t_i}\alpha\right) = 0$$

for all $\alpha \in H^0(C, \Omega_C) \cong k^g$.

For example: $C = \mathrm{Proj}\ k[x,y,z]/(x^3+y^3+z^3)$

$$H^0(C, \Omega_C) \cong k \cdot \frac{z^2}{y^2} \cdot d\left(\frac{x}{z}\right)$$

$\rightsquigarrow$ explicit linear condition on principal parts at any $p_1, \ldots, p_d \in C$.