# PROBLEM SET 6 (DUE ON TUESDAY, MARCH 14)

(All Exercises are references to the December 31, 2022 version of *Foundations of Algebraic Geometry* by R. Vakil.)

**Problem 1.** Exercise 19.6.J (Bezout's Theorem - if you want, you can just consider the case where $X$ is reduced, or you can learn a little about associated points for the general case)

**Problem 2.** Suppose $Z_1, Z_2$ are closed subschemes of a projective $k$-scheme $X$. Show that

$$\chi(Z_1 \cup Z_2, \mathcal{O}_{Z_1 \cup Z_2}) = \chi(Z_1, \mathcal{O}_{Z_1}) + \chi(Z_2, \mathcal{O}_{Z_2}) - \chi(Z_1 \cap Z_2, \mathcal{O}_{Z_1 \cap Z_2}),$$

where $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are the scheme-theoretic union and intersection of closed subschemes. (In fact, Hilbert polynomials computed with respect to a (very) ample line bundle on the ambient scheme $X$ will satisfy the same identity - compare with Exercise 19.6.O.)

**Problem 3.** Let $X \subset \mathbb{P}^3_k$ be the union of three distinct lines that all pass through a single point. Is this enough information to determine the arithmetic genus of $X$, or do you need to know what the three lines are?

**Problem 4.** Exercise 19.6.R (arithmetic genus of a complete intersection of surfaces in $\mathbb{P}^3$ - in this context complete intersection just means that the surfaces being intersected do not share an irreducible component)

**Problem 5.** Let $m, n$ be positive integers and let $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m,n))$ be a bihomogeneous polynomial of degree $(m,n)$. (Here $\mathcal{O}(m,n) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(m) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(n)$, where $\pi_1, \pi_2$ are the two projections.) Compute the arithmetic genus of the curve $X = V(f)$. Your answer for $(m,n) = (d,d)$ should agree with your answer to the previous problem when $(m,n) = (2,d)$ - why is this? (Hint: use the ideal sheaf sequence for $X$ inside $\mathbb{P}^1 \times \mathbb{P}^1$.)

**Problem 6.** Show that for $n \geq 3$, the intersection of any two hypersurfaces in $\mathbb{P}^n_k$ is connected. (Hint: if $X$ is the intersection, compute $h^0(X, \mathcal{O}_X) = 1$ using two long exact sequences of cohomology groups.)