PROBLEM SET 10 (POSTED ON THURSDAY, APRIL 2)

(All Exercises are references to the October 21, 2025 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 19.11.B (a map of projective varieties not arising from a map of (regraded) graded rings)

Problem 2. Suppose (E, p) is a (smooth) elliptic curve in Weierstrass form $y^2 = x^3 + ax + b$ (along with the point at infinity p, so really $E = \operatorname{Proj} k[x, y, z]/(y^2z - x^3 - axz^2 - bz^3)$). Show that the rational section $(dx)/y$ of $\Omega_{E/k}$ actually has no zeroes or poles in E, so it generates $\Omega_{E/k} \cong \mathcal{O}_E$.

Problem 3. Let $\iota : C \to \mathbb{P}^2_k$ be the inclusion of a smooth plane curve of degree $d > 0$. Let $\mathcal{F} = \iota^*\Omega_{\mathbb{P}^2/k}$, a rank 2 vector bundle on C. Compute the Euler characteristic $\chi(C, \mathcal{F})$.

Problem 4. Let $X = \operatorname{Bl}_{(0,0)} \mathbb{A}^2_k$ be the blow-up of the affine plane at the origin. Compute the canonical bundle K_X. (Recall that $\operatorname{Pic}(X)$ is generated by $\mathcal{O}_X(E)$, so the question is for which $d \in \mathbb{Z}$ we have $K_X \cong \mathcal{O}_X(dE)$.)

Problem 5. Let X be a regular degree d surface in $\mathbb{P}^3_{\mathbb{C}}$. Compute $h^0(X, K_X)$. This is called the *geometric genus* of the surface X (see Exercise 21.5.C). (Hint: use adjunction to compute K_X as the restriction of a line bundle on $\mathbb{P}^3$, then pushforward to $\mathbb{P}^3$ and compute cohomology there using a long exact sequence and the known values of $h^i(\mathbb{P}^3, \mathcal{O}(m))$.)