PROBLEM SET 8 (POSTED ON THURSDAY, MARCH 12)

(All Exercises are references to the October 21, 2025 version of *Foundations of Algebraic Geometry* by R. Vakil.)

Problem 1. Exercise 18.6.J (Bezout's Theorem - if you want, you can just consider the case where X is reduced, or you can learn a little about associated points for the general case)

Problem 2. Suppose Z_1, Z_2 are closed subschemes of a projective k-scheme X. Show that

$$\chi(Z_1 \cup Z_2, \mathcal{O}_{Z_1 \cup Z_2}) = \chi(Z_1, \mathcal{O}_{Z_1}) + \chi(Z_2, \mathcal{O}_{Z_2}) - \chi(Z_1 \cap Z_2, \mathcal{O}_{Z_1 \cap Z_2}),$$

where $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are the scheme-theoretic union and intersection of closed subschemes. (In fact, Hilbert polynomials computed with respect to a (very) ample line bundle on the ambient scheme X will satisfy the same identity - compare with Exercise 18.6.O.)

Problem 3. Let $X \subset \mathbb{P}^3_k$ be the union of three distinct lines that all pass through a single point. Is this enough information to determine the arithmetic genus of X, or do you need to know what the three lines are?

Problem 4. Exercise 18.6.R (arithmetic genus of a complete intersection of surfaces in $\mathbb{P}^3$ - in this context complete intersection just means that the surfaces being intersected do not share an irreducible component)

Problem 5. Let m, n be positive integers and let $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m, n))$ be a bihomogeneous polynomial of degree (m, n). (Here $\mathcal{O}(m, n) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(m) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(n)$, where π_1, π_2 are the two projections.) Compute the arithmetic genus of the curve $X = V(f)$. Your answer for $(m, n) = (d, d)$ should agree with your answer to the previous problem when $(m, n) = (2, d)$ - why is this? (Hint: use the ideal sheaf sequence for X inside $\mathbb{P}^1 \times \mathbb{P}^1$.)

Problem 6. Show that for $n \geq 3$, the intersection of any two hypersurfaces in $\mathbb{P}^n_k$ is connected. (Hint: if X is the intersection, compute $h^0(X, \mathcal{O}_X) = 1$ using two long exact sequences of cohomology groups.)