PROBLEM SET 1 SOLUTIONS

(These solutions are just meant to explain how I think about each of the problems personally - there were many ways to think about some of these problems, and many choices to be made in writing up solutions.)

Problem 1. Any genus 0 curve is isomorphic to $\mathbb{P}^1$, so

$$M_{0,n} = \{(x_1, \dots, x_n) \in (\mathbb{P}^1)^n \mid x_i \neq x_j\} / \operatorname{Aut}(\mathbb{P}^1).$$

The automorphism group of $\mathbb{P}^1$, PGL_2, acts strictly 3-transitively, so this is isomorphic to

$$\{(x_1, \dots, x_n) \in (\mathbb{P}^1)^n \mid x_1 = 0, x_2 = 1, x_3 = \infty, x_i \neq x_j\},$$

which is visibly isomorphic to an open subset of $\mathbb{C}^{n-3}$ (it is the complement of a bunch of hyperplanes), which itself is an open subset of $\mathbb{P}^{n-3}$.

The transposition (13) acts via the S_n-action on this open subset by the Cremona transformation $[z_0 : \cdots : z_{n-3}] \mapsto [z_0^{-1} : \cdots : z_{n-3}^{-1}]$, which does not extend to the locus where two or more of the z_i are zero. This shows that compactifying $M_{0,n}$ to $\mathbb{P}^{n-3}$ in this way does not handle the n marked points symmetrically for $n \geq 5$.

Problem 2. Let S be an oriented torus (as a topological space) with a marked point p and oriented loops $e, f \in \pi_1(S, p) = H_1(S; \mathbb{Z})$ generating the integral homology of S, with $e \cdot f = 1$. Suppose that (X, q) is a Riemann surface with a given orientation-preserving homeomorphism from (S, p). The universal cover of X is isomorphic to $\mathbb{C}$ (as a Riemann surface) by uniformization, and this isomorphism is unique up to affine transformations $z \mapsto az + b$. Moreover, the deck transformations on this universal cover correspond to translations on $\mathbb{C}$. Let τ_e, τ_f be the translation vectors for the deck transformations corresponding to the loops e, f; these are well-defined up to scaling both by a single element of $\mathbb{C}^*$. By the orientation data $e \cdot f = 1$, we see that τ_e / τ_f is a (well-defined) element of the upper half-plane $\mathbb{H}$, and this defines a map $T_{1,1} \to \mathbb{H}$.

The mapping class group $\mathrm{Mod}_{1,1}$ can be identified with $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_2(\mathbb{Z})$ via the symplectic representation defined by acting on the first homology of S. The action of $\mathrm{Mod}_{1,1}$ on $T_{1,1}$ (defined as a right action) is given by composition of a self-homeomorphism of S with the homeomorphism between S and X, so the map $e \mapsto ae + cf$, $f \mapsto be + df$ will send $\xi = \tau_e / \tau_f$ to $(a\tau_e + c\tau_f)/(b\tau_e + d\tau_f) = (a\xi + c)/(b\xi + d)$, which is indeed a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. (You might have gotten the inverse of this if you set things up to have a left action, and you might have been composed with a conjugation automorphism of $\mathrm{SL}_2(\mathbb{Z})$ if you chose a different ordering on your basis - the precise details here aren't that canonical.)

Problem 3. Fix a topological genus 2 surface S. The isotopy class of the hyperelliptic involution on a genus 2 Riemann surface then defines a map from $T_2 = T(S)$, the Teichmuller space of isotopy classes of S-marked Riemann surfaces, to $\mathrm{Mod}_2 = \mathrm{Mod}(S)$, the mapping class group of isotopy classes of self-homeomorphisms of S. This map $\phi : T_2 \to \mathrm{Mod}_2$ clearly respects the action of Mod_2 on T_2, in the sense that $\phi(xg) = g^{-1}\phi(x)g$ for any $x \in T_2$ and $g \in \mathrm{Mod}_2$. Thus it suffices to show that ϕ is a constant map, since then the image element will be central (and non-trivial since the hyperelliptic involution acts non-trivially on first homology). But ϕ is clearly continuous (e.g. this can be checked locally using the hyperelliptic description of a genus 2 curve as a double cover of P^1 ramified at 6 points) and T_2 is contractible, hence connected, while Mod_2 is a discrete group, so ϕ is indeed constant.

Problem 4. Computation 1: The forgetful map $M_{2,1} \to M_2$ is an orbifold fiber bundle with fiber a surface X of genus 2 (informally this means that the fiber over a point with orbifold structure group G is X/G; formally, this means that the bundle is locally the quotient of a topological fiber bundle by a finite group acting on both the base and the total space). This means that $\chi_{\mathrm{orb}}(M_{2,1}) = \chi(X)\chi_{\mathrm{orb}}(M_2)$ (this multiplicativity follows from the multiplicativity for topological fiber bundles upon taking quotients), so $\chi_{\mathrm{orb}}(M_2) = (1/120)/(-2) = -1/240$.

Computation 2: The fact that any genus 2 curve is hyperelliptic via a unique hyperelliptic involution, combined with the fact that there is a unique double cover of $\mathbb{P}^1$ ramified over any set of six points, implies that M_2 and $M_{0,6}/S_6$ are homeomorphic as toplogical spaces. But as orbifolds they are not isomorphic - the orbifold structure on M_2 has an extra factor of 2 everywhere thanks to the hyperelliptic involution. (One way to think about this: the orbifold Euler characteristic of M_2 is defined via $M_2 = T_2/\,\mathrm{Mod}_2$, but the hyperelliptic involution in Mod_2 actually acts trivially on T_2 so it affects the orbifold structure of M_2 as well as its orbifold Euler characteristic without affecting the topology.) The result is that $\chi_{\mathrm{orb}}(M_2) = \chi_{\mathrm{orb}}(M_{0,6}/S_6)/2 = \chi(M_{0,6})/1440$. It remains to compute the Euler characteristic of $M_{0,6}$, which can be done by using the fact that $M_{0,6}$ is the total space of a fiber bundle over $M_{0,5}$ with fiber a sphere with five punctures, and so on: $\chi(M_{0,6}) = (-3)(-2)(-1) = -6$. Thus $\chi_{\mathrm{orb}}(M_2) = -6/1440 = -1/240$.

Problem 5. Solution 1 (group cohomology): (We do the computation in cohomology with integer coefficients - sorry for any confusion caused by using $\mathbb{Q}$-coefficients in the problem statement, but the answer being zero with integer coefficients implies that it will also be zero with $\mathbb{Q}$-coefficients.) The class ψ (up to sign) corresponds to the central extension of $\mathrm{Mod}_{g,1}$ given by the capping exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Mod}_g^1 \to \mathrm{Mod}_{g,1} \to 1.$$

Call the first (nontrivial) map in this sequence f; the second map is the capping homomorphism, called c. We want to pull back this class by c, so the result will correspond to the pullback of the central extension:

$$1 \to \mathbb{Z} \to \mathrm{Mod}_g^1 \times_{\mathrm{Mod}_{g,1}} \mathrm{Mod}_g^1 \to \mathrm{Mod}_g^1 \to 1,$$

where the middle group in the sequence is the fiber product of groups, the first map is $(f, 0)$, and the second map is the projection to the second component. This extension has an obvious section coming from the diagonal homomorphism $\mathrm{Mod}_g^1 \to \mathrm{Mod}_g^1 \times_{\mathrm{Mod}_{g,1}} \mathrm{Mod}_g^1$, so it splits and hence corresponds to the zero class in cohomology.

Solution 2 (topology): We think about the surface bundle classifying spaces BMod_g^1 and $\mathrm{BMod}_{g,1}$ and their universal bundles. Let S_g^1 be the universal bundle on BMod_g^1 (so the fibers are surfaces with one boundary component, and the bundle is a trivial circle bundle when restricted to the boundary components of the fibers), and let $S_{g,1}$ be the universal bundle on $\mathrm{BMod}_{g,1}$ (i.e. a surface bundle with a given section). Then a map $\mathrm{BMod}_g^1 \to \mathrm{BMod}_{g,1}$ that induces the capping homomorphism on fundamental groups must have the property that the pullback of $S_{g,1}$ is isomorphic to the surface bundle given by capping off the boundary of S_g^1

(and taking a section given by a marked point in the disc that you use to cap it). Then the vertical tangent bundle to $S_{g,1}$ pulls back to (something isomorphic to) the vertical tangent bundle to the capped S_g^1, which is the vertical tangent bundle to a trivial disc bundle, hence a trivial bundle. Therefore the pullback of ψ, the first Chern class of the vertical tangent bundle to $S_{g,1}$, is the first Chern class of a trivial bundle, hence zero.