PROBLEM SET 1 (DUE TUESDAY, OCT 25)

If you have any questions about any of the problems (either clarifications about what I mean, or concerns about not having the right technical knowledge to think about the problem properly), please let me know. In general, the problems are designed to give practice with different ways of thinking about moduli spaces of curves rather than to challenge your technical knowledge, and you should worry more about understanding what is going on than about having precise justifications for everything.

Problem 1. Let $n \geq 3$. Describe a way to identify $M_{0,n}$ (i.e. the configuration space of n distinct points $p_1, \ldots, p_n$ on the sphere $S^2 = \mathbb{P}^1$, up to isomorphism) with an open subspace of complex projective space $\mathbb{P}^{n-3}$. Note that the symmetric group S_n has a natural action on $M_{0,n}$ by permuting the indices on the n marked points. When $n = 4$, you can check (if you want) that this action extends to an action on $\mathbb{P}^1$ (by elements of PGL_2). Does this continue for general n? What does this suggest about whether the standard compactification $\overline{M}_{0,n}$ (which we haven't defined yet) should be isomorphic to $\mathbb{P}^{n-3}$?

Problem 2. We saw in class that the simplicial complex "$A \setminus A_\infty$" construction of the Teichmuller space $T_{1,1}$ produces in the end something that looks like hyperbolic space. Give a direct identification of $T_{1,1}$ with the upper half-plane coming from the definition of $T_{1,1}$ via S-marked Riemann surfaces (with a marked point). Also identify the mapping class group $\mathrm{Mod}_{1,1}$ with $\mathrm{SL}_2(\mathbb{Z})$, and then describe what the action of $\mathrm{Mod}_{1,1}$ on $T_{1,1}$ looks like under these identifications.

Problem 3. I've mentioned in class a couple times the fact that every genus 2 algebraic curve (or Riemann surface, or hyperbolic surface...) has a unique hyperelliptic involution, i.e. a degree 2 automorphism with quotient of genus 0. (If you want, you can use a pants decomposition of a genus 2 hyperbolic surface to construct such an involution, though proving uniqueness is trickier.) Assuming this fact (both existence and uniqueness), show that the group Mod_2 has nontrivial center.

Problem 4. Compute the orbifold Euler characteristic of M_2 in two ways and check that you get the same answer. (The two that I would suggest are: (1) Use the Harer-Zagier theorem to get that the orbifold Euler characteristic of $M_{2,1}$ is $\zeta(-3) = 1/120$, and then relate $M_{2,1}$ to M_2; (2) Use the fact that all genus 2 curves are hyperelliptic to relate M_2 to $M_{0,6}$, and then compute the (regular) Euler characteristic of $M_{0,6}$ (e.g. by relating it to $M_{0,3}$). In both cases, it will be useful that Euler characteristic is multiplicative in fiber bundles. If your two computations aren't quite matching up, you probably need to think more about stabilizers!)

Problem 5. Recall that the capping homomorphism on mapping class groups replaces a boundary component with a marked point, by gluing in a disc containing a marked point. This homomorphism induces a map on cohomology rings

$$c^* : H^*(\mathrm{Mod}_{g,1}; \mathbb{Q}) \to H^*(\mathrm{Mod}_{g,[1]}; \mathbb{Q}).$$

Recall that we considered a class ψ (or $-e$) in $H^2(\mathrm{Mod}_{g,1}; \mathbb{Q})$. Compute the image $c^*\psi$ under the above map. (If you know (or are willing to learn) a little group cohomology, then you should be able to do this easily using the central extension definition of ψ. If not, recall that ψ is the Euler class of a vector bundle formed by taking the cotangent space at the marked point, and think about how that bundle behaves after being pulled back by the capping homomorphism.)