PROBLEM SET 1 (DUE ON SEP 21)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Exercise 2.20 (extension of scalars preserves flatness - note this exercise is in the middle of the chapter, not at the end like most of them)

Problem 2. Chapter 2, Exercise 6 (polynomials with coefficients in a module)

Problem 3. Observe that $A/I \otimes_A A/I \cong A/I$ for any ideal $I \subseteq A$ and $S^{-1}A \otimes_A S^{-1}A \cong S^{-1}A$ for any mult. closed $S \subseteq A$. (Think about these two facts if they aren't immediately clear!) Give an example of a ring A and a module M such that $M \otimes_A M \cong M$, but where M is not isomorphic to either of the preceding examples (as A-modules). In other words, if you choose $A = \mathbb{Z}$ then you must give an example of a $\mathbb{Z}$-module M that satisfies $M \otimes_{\mathbb{Z}} M \cong M$ but is not isomorphic to $\mathbb{Z}/I$ or $S^{-1}\mathbb{Z}$ for any I or S.

Problem 4. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ as rings. (Hint: try to find two different ring homomorphisms $f, g : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$, then check whether $(f, g) : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}^2$ is an isomorphism.)

Problem 5. Chapter 3, Exercise 3 (composition of localizations)

Problem 6. Let A be a ring. An element $x \in A$ is called a *zero-divisor* if there exists $y \in A$ with $y \neq 0$ and $xy = 0$. Let $S_0 \subset A$ be the set of non-zero-divisors. Prove that every element of $S_0^{-1}A$ is either a unit or a zero-divisor. (Note: this is part of Chapter 3, Exercise 9. The ring $S_0^{-1}A$ is called the *total ring of fractions* of A.)

Problem 7. Let A be a ring. Show that any localization $S^{-1}A$ of A is isomorphic to a subring of the total ring of fractions of A/I for some ideal I.