PROBLEM SET 10 (DUE ON NOV 30)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Chapter 10, Exercise 4 (zero-divisors in completions)

Problem 2. Let A be a Noetherian ring and let $I \subseteq A$ be an ideal. Let $\widehat{A}$ be the I-adic completion of A. Show that $\widehat{A}$ is isomorphic to a quotient of the ring of formal power series $A[[t_1, \ldots, t_k]]$ for some $k \geq 0$. (Hint: choose finitely many generators $a_1, \ldots, a_k$ for I. Try to define a surjective ring homomorphism $f : A[[t_1, \ldots, t_k]] \to \widehat{A}$ with $f(t_i) = a_i$.)

Problem 3. Let A be a DVR with maximal ideal $\mathfrak{m}$. Show that the $\mathfrak{m}$-adic completion $\widehat{A}$ of A is also a DVR. (Hint: let v be the valuation on A and define a map $\widehat{v} : \widehat{A} \to \mathbb{Z}$ by $\widehat{v}((a_n)) = v(a_n)$ for n taken large enough such that $a_n \neq 0 \in A/\mathfrak{m}^n$ (and checking that this makes sense). Use $\widehat{v}$ to show that $\widehat{A}$ is a domain, and then show that it extends to a discrete valuation on the field of fractions of $\widehat{A}$.)

Problem 4. Let A be a Dedekind domain and let $\mathfrak{m}$ be a maximal ideal of A. Show that the $\mathfrak{m}$-adic completion of A is a DVR. (Hint: use the previous problem.)

Problem 5. By the previous problem, the ring of p-adic integers $\mathbb{Z}_p$ (the completion of $\mathbb{Z}$ at (p)) is a DVR. The field of fractions of $\mathbb{Z}_p$ is denoted $\mathbb{Q}_p$ and called the *p-adic rationals*. Prove that the automorphism group of $\mathbb{Q}_p$ is trivial, i.e. the only ring isomorphism $\mathbb{Q}_p \to \mathbb{Q}_p$ is the identity map. (Hint: for any positive integer k, let $S_k \subseteq \mathbb{Q}_p$ be the set of kth powers in $\mathbb{Q}_p$. By intersecting some of the S_k together (and using a combination of the p-adic valuation on $\mathbb{Q}_p$ and Hensel's lemma on $\mathbb{Z}_p$ to compute these S_k), show that the subset $1 + p\mathbb{Z}_p$ must be preserved by any automorphism. Then the subset $a + p^n\mathbb{Z}_p$ must be preserved for any integers a, n, and you should be able to finish from there.)