PROBLEM SET 11 (DUE ON DEC 12)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Let A be a Noetherian ring with $\dim A \geq 2$. Show that A has infinitely many prime ideals. (Hint: Use Krull's principal ideal theorem repeatedly to construct prime ideals of height ≤ 1.)

Problem 2. Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be Noetherian local rings and let $\phi : A \to B$ be a ring homomorphism with $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$. Show that

$$\dim B \leq \dim A + \dim B/\phi(\mathfrak{m})B.$$

(Hint: Try to construct an $\mathfrak{n}$-primary ideal in B with the appropriate number of generators.)

Problem 3. Let A be a Noetherian ring. Show that $\dim A[t] = \dim A + 1$. (Hint: suppose that $\mathfrak{q} \subset A[t]$ is a maximal ideal and take $\mathfrak{p} = \mathfrak{q} \cap A \subset A$. Then apply the previous problem to the map $A_\mathfrak{p} \to A[t]_\mathfrak{q}$. You can assume without proof that $\dim k[t] = 1$ for any field k, since we basically proved that in class.)

Problem 4.
(a) Chapter 11, Exercise 1 (non-singular points on a hypersurface - we will discuss regular local rings on Tuesday, so you may want to wait until then to do this problem and Problem 5)
(b) Let $A = \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2)$. Let $\mathfrak{m} = (x, y, z) \subset A$, a maximal ideal. Compute $\dim_\mathbb{C} A/\mathfrak{m}^n$ as a polynomial in n.

Problem 5. Let A be a ring. A sequence of elements $a_1, \ldots, a_d \in A$ is called a *regular sequence* if the ideal $(a_1, \ldots, a_d)$ is proper and for each $i = 1, \ldots, d$, the image of a_i in $A/(a_1, \ldots, a_{i-1})$ is not a zero-divisor. Show that if (A, m) is a regular local ring of dimension d and $m = (a_1, \ldots, a_d)$, then $a_1, \ldots, a_d$ is a regular sequence. (Hint: Use the fact that a regular local ring is a domain. We will discuss regular local rings on Tuesday.)