PROBLEM SET 3 (DUE ON OCT 5)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Chapter 5, Exercise 2 (homomorphisms into an algebraically closed field can be extended along integral ring extensions)

Problem 2. Chapter 5, Exercise 3 (tensor products preserve integrality)

Problem 3. Chapter 5, Exercise 4 (integrality is not in general preserved by localizing at prime ideals - compare with Proposition 5.6(ii), which is the correct sense in which localization preserves integrality)

Problem 4. Chapter 5, Exercise 7 (conditions under which a subring is integrally closed)

Problem 5. (a) Let A be a ring. Let G be a finite group along with an action on A via ring automorphisms (i.e. there is a map $G \times A \to A, (g, a) \mapsto g \cdot a$ satisfying $g \cdot (h \cdot a) = (gh) \cdot a$ and such that $a \mapsto g \cdot a$ is an automorphism of A for each $g \in G$). Let $A^G \subseteq A$ be the subring of elements fixed by every $g \in G$. Prove that A is integral over A^G. (This is the first part of Chapter 5, Exercise 12.)

(b) Now let $A = \mathbb{C}[x_1, \ldots x_n]$ and take G to be the symmetric group S_n, acting on A by permuting the variables. Find all prime ideals $\mathfrak{p} \subset A$ lying over the prime ideal $(x_1 x_2 \cdots x_n) \subset A^G$.

Problem 6. Let $A = \mathbb{C}[x, y]/(y^2 - x^3)$. Let $B = A_{(0)}$ be the field of fractions of A. Determine the integral closure C of A in B. Show that the inclusion $A \subset C$ induces a bijection $\operatorname{Spec} C \to \operatorname{Spec} A$. (In other words, each prime ideal in A has exactly one prime ideal lying above it in C.)