PROBLEM SET 4 (DUE ON OCT 12)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Let k be an algebraically closed field and let $n \geq 1$. Let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ be a collection of polynomials such that for $y = (y_i)_{i=1}^n \in k^n$, $f_1(y) = f_2(y) = \cdots = f_m(y) = 0$ implies that $y_i = 0$ for all i. Show that there exists a positive integer N such that the ideal $(x_1, \dots, x_n)^N$ is contained in the ideal $(f_1, \dots, f_m)$.

Problem 2. Chapter 6, Exercise 1, part (i) (a surjective homomorphism from a Noetherian module to itself is an isomorphism)

Problem 3. Let M, N be Noetherian A-modules. Show that $M \otimes_A N$ is also a Noetherian A-module. (Hint: you may find the following stronger result easier to prove: if M is f.g. and N is Noetherian, then $M \otimes_A N$ is Noetherian.)

Problem 4. In a composition series $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_l = M$, the simple modules M_{i+1}/M_i are called the *composition factors*. Let N be a maximal proper submodule of M (so M/N is simple). Show that any composition series of M must have a composition factor that is isomorphic to M/N (as A-modules). (Hint: take i maximal such that $M_i \subseteq N$. Note: with a little more work you can actually prove the *Jordan-Hölder theorem*: the multiset of isomorphism classes of composition factors does not depend on the choice of composition series. We just need this weaker fact for the following problem though.)

Problem 5. Let A be a ring that is both Noetherian and Artinian. Show that A has only finitely many maximal ideals. (Hint: you can show that the A-modules A/m for m maximal are not isomorphic for distinct m, and A admits a composition series as a module over itself. Then use the previous problem.)

Problem 6. Let k be a field. Let A be a finite k-algebra. Show that A has only finitely many prime ideals. (Hint: Prove that every prime ideal in A is maximal and that A is both Noetherian and Artinian. Then use the previous problem.)

Problem 7. Let $f : A \to B$ be a ring homomorphism with B finite over A. Let $\mathfrak{p} \subset A$ be a prime ideal. Show that there are only finitely many prime ideals $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$, i.e. with $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. (Hint: Reduce to the previous problem using appropriate localizing and quotienting.)