PROBLEM SET 6 (DUE ON OCT 26)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Let k be a field and $n \geq 0$. Let $A = k[x_1, \ldots, x_n]$. An ideal $I \subseteq A$ is called *monomial* if it is generated by elements of the form $x_1^{k_1} \cdots x_n^{k_n}$ with $k_1, \ldots, k_n \geq 0$. Determine which monomial ideals $I \subseteq A$ are primary. (For example, the ideal $(x_1^2, x_1 x_2)$ discussed in class is monomial but is not primary.)

Problem 2. With k, n, A as in the previous problem, now determine which monomial ideals $I \subseteq A$ are irreducible (recall a proper ideal I is irreducible if and only if we do not have $I = J \cap K$ with J, K both strictly larger than I).

For the remaining problems, we will recall the definition of $\operatorname{Ass}_A(M)$ given in class (and in Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*) and the basic properties of it that were proved in class. You may use the following properties in the problems, but please do not use stronger results about associated primes from Eisenbud or elsewhere.

Definition. If M is an A-module, the set of *associated primes* of M is

$$\operatorname{Ass}_A(M) := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} = \operatorname{Ann}(v) \text{ for some } v \in M\}.$$

Properties:

- If A is Noetherian and $M \neq 0$, then $\operatorname{Ass}_A(M) \neq \emptyset$.
- If $M \subseteq N$, then $\operatorname{Ass}_A(M) \subseteq \operatorname{Ass}_A(N)$.
- $\operatorname{Ass}_A(M \oplus N) = \operatorname{Ass}_A(M) \cup \operatorname{Ass}_A(N)$.
- For I an ideal in a Noetherian ring A,

$$\operatorname{Ass}_A(A/I) = \{\mathfrak{p}\} \Leftrightarrow I \text{ is } \mathfrak{p}\text{-primary}.$$

- If A is Noetherian and $M_0, \ldots, M_n$ are submodules of an A-module M satisfying
  (a) $M_0 = M_1 \cap \cdots \cap M_n$,
  (b) $M_0 \neq M_1 \cap \cdots \cap M_{j-1} \cap M_{j+1} \cap \cdots \cap M_n$ for all $1 \leq j \leq n$,
  (c) $\operatorname{Ass}_A(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leq i \leq n$,
  then $\operatorname{Ass}_A(M/M_0) = \{\mathfrak{p}_i \mid 1 \leq i \leq n\}$.

Problem 3. Give an example of a ring A and a nonzero A-module M with no associated primes. (Hint: A must be non-Noetherian, and $\mathbb{C}[x_1, x_2, \ldots]$ is a good non-Noetherian ring to think about.)

Problem 4. Let A be a Noetherian ring. Let M be an A-module. Let $\mathfrak{p} \subset A$ be a prime ideal such that there exists an element $v \in M$ with $\operatorname{Ann}(v) \subseteq \mathfrak{p}$. Show that there exists an associated prime $\mathfrak{p}' \in \operatorname{Ass}_A(M)$ with $\mathfrak{p}' \subseteq \mathfrak{p}$. (Hint: let S be the set of annihilators (of elements of M) that are contained in $\mathfrak{p}$ and take a maximal element $I = \mathrm{Ann}(v) \in S$. If I is not prime, get a contradiction by constructing a suitable element $a \in A$ that gives a larger ideal $\mathrm{Ann}(av) \in S$.)

Problem 5. Let A be a Noetherian ring. Let I be an ideal in A. Let $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be a minimal primary decomposition of I. Let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, so $\mathrm{Ass}_A(A/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Suppose that $\mathfrak{p}_1$ is minimal among the $\mathfrak{p}_i$, i.e. $\mathfrak{p}_i \subseteq \mathfrak{p}_1 \implies i = 1$. Prove that

$$\mathfrak{q}_1 = \{a \in A \mid ab \in I \text{ for some } b \in A \setminus \mathfrak{p}_1\}$$

and thus does not depend on the choice of minimal primary decomposition. (Hint: use the previous problem to show that the ideals I and $\mathfrak{q}_1$ are equal after localizing at $\mathfrak{p}_1$.)